Delay-Adaptive Predictor Feedback Control of Reaction–Advection–Diffusion PDEs With a Delayed Distributed Input

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***Abstract*—We consider a system of reaction–advection–diffusion partial differential equation (PDE) with a distributed input subject to an unknown and arbitrarily large time delay. Using Lyapunov technique, we derive a delay-adaptive predictor feedback controller to ensure the global stability of the closed-loop system in the L^2 sense. More precisely, we express the input delay as a 1-D transport PDE with a spatial argument leading to the transformation of the time delay into a spatially distributed shift. For the resulting mixed transport and reaction–advection–diffusion PDE system, we employ a PDE backstepping design and certainty equivalence principle to derive the suitable adaptive control law that compensates for the effect of the unknown time delay. Our controller ensures the global stabilization in the L^2 sense. Our result is the first delay-adaptive predictor feedback controller with a PDE plant subject to a delayed distributed input. The feasibility of the proposed approach is illustrated by considering a mobile robot that spread a neutralizer over a polluted surface to achieve efficient decontamination with an unknown actuator delay arising from the noncollocation of the contaminant diffusive process and the moving neutralizer source. Consistent simulation results are presented to prove the effectiveness of the proposed approach.**

***Index Terms*—Delay-adaptive control, distributed input delay, partial differential equation (PDE) backstepping, predictor feedback, reaction–advection–diffusion PDE.**

I. Introduction

Interactions of local quantities and local reaction kinetics governed by physical laws are often described by reaction–advection–diffusion equations. The occurrence of diffusive phenomena in diverse physical processes involving chemical reactions [1], thermal fluids [2], biological pattern formation [3], etc., gives rise to various challenges. Over the past few decades, these challenges motivated the development of many control design approaches to enable stabilization among other objectives.

Relevant advances have been achieved toward the controlling of diffusion-driven distributed parameter systems with boundary actuation. Early controllers are constructed upon reduced-order models that approximate the infinite-dimensional systems by finite-dimensional ones. However, the stability and performance of such design should be validated for the original partial differential equation (PDE), or at least a high-order approximation of it to avoid instability [4]. As more recent results, various infinite-dimensional control design techniques that are based on PDE dynamics have emerged, among which, the popular backstepping design allows exponential stabilization of plants with unstable reaction terms [5], [6]. As well, robust regulation for systems subject to unmeasurable in-domain and boundary disturbances, which are described by a finite-dimensional signal model, has been achieved via backstepping design [7]. Later on, backstepping has been exploited for the control of coupled reaction–diffusion systems with constant and spatially varying coefficients [8]–[10]. The key idea of the backstepping design relies on the choice of an invertible Volterra transformation, which maps an unstable plant into a target system whose stability can be derived from a Lyapunov argument.

On the other hand, the study of dynamical systems governed by reaction–diffusion PDEs subject to actuators delays that are deleterious to stability has become an active research topic since the pioneering contribution [11], which develops a PDE backstepping boundary controller to compensate the delay effect. The aforementioned method has been extended to a 3-D formation control problem to compensate for the effect of potential input delays [12]. Alternatively, a series expansion approach and a Lyapunov-based point control approach design have been proposed in [13] and [14], respectively. In the context of a networked architecture that can remotely actuate a reaction–diffusion process with sampled point measurements and a delayed point actuation, the works in [14]–[16] developed efficient control algorithms using Lyapunov–Krasovskii functionals to derive LMI-based stability conditions (see also Karafyllis and Krstic [17], which deal with sampled-data control of PDEs). For a relatively small unknown input delay, Katz and Fridman [18] develop a finite-dimensional observer-based controller for a reaction–diffusion PDE by using a modal decomposition approach whose stability is analyzed by Lyapunov functionals combined with Halanay's inequality.

In this article, we design a delay-adaptive controller for a one-dimensional reaction–advection–diffusion PDE with arbitrarily large unknown delayed distributed input. The key point of our approach lies in the conversion of the input delay into a transport PDE containing a spatial argument. Here, the spatial argument transforms the time-delay into an in-domain spatially distributed shift [19], [20] resulting in a mixed system different from the PDE–PDE cascade system introduced in [11]. We employ the PDE backstepping method and the choice of the unknown delay parameter's update law leads to a target system

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structured as an "in-domain cascade system" whose L^2 global stability is established using suitable Lyapunov functionals. The invertible Volterra/backstepping transformation enables to establish the norm equivalence between the target system and the plant resulting in the L^2 global stability of the PDE plant subject to the designed delay-adaptive compensated controller. It is worth to mention that for delay distributed systems, controllers to compensate input delays based on predictor feedback approach and backstepping method can be found in [11] and [21]–[28]. *This is possibly the first time a delay-adaptive control method has been applied to a PDE plant with a distributed input.*

Following the works in [29]–[32], our control design is applied to a simplified model of surface decontamination process involving a remote moving source that sprays a decontaminant or neutralizer on a surface charged with diffusive toxic particles. We account for an unknown input delay induced by the transport of the neutralizer from the noncollocated source to the contaminated surface governed by a reaction–diffusion PDE.

This article is organized as follows. Section II briefly describes the design of a nonadaptive controller for the considered reaction–advection–diffusion PDE system. In Section III, the design of the delay-adaptive control law is discussed. Section IV is dedicated to the stability analysis of the resulting adaptive closed-loop system, and the decontamination process is described in Section V with consistent simulation results proving the feasibility of our approach.

Notation: Throughout this article, we adopt the following notation to define the L^2-norm for $\chi(\cdot) \in L^2[-\ell,\ell]$ and $\phi(\cdot,\cdot) \in L^2([-\ell,\ell] \times [0,1])$:

$$\|\chi\|_{L^2}^2 = \int_{-\ell}^{\ell} |\chi(x)|^2 \mathrm{d}x, \quad \|\phi\|_{L^2}^2 = \int_{-\ell}^{\ell}\int_0^1 |\phi(x,s)|^2 \mathrm{d}s\mathrm{d}x \tag{1}$$

and set $\|\chi\|^2 = \|\chi\|_{L^2}^2$ and $\|\phi\|^2 = \|\phi\|_{L^2}^2$. Also define

$$H_E^1[-\ell,\ell] = \{\chi(\cdot) \in H^1(-\ell,\ell), \chi(-\ell) = \chi(\ell) = 0\} \tag{2}$$

$$L_E^2([-\ell,\ell] \times [0,1]) = \{\phi(\cdot,\cdot) \in L^2([-\ell,\ell] \times [0,1)) \; \phi(\cdot,1) = 0\}. \tag{3}$$

For any given function $\psi(\cdot,\hat{D}(t))$

$$\frac{\partial\psi(\cdot,\hat{D}(t))}{\partial t} = \dot{\hat{D}}(t)\frac{\partial\psi(\cdot,\hat{D}(t))}{\partial\hat{D}(t)}. \tag{4}$$

II. Problem Statement and Nonadaptive Controller

Consider the reaction–advection–diffusion PDE with a known distributed actuator delay D defined as follows:

$$u_t(x,t) = \varepsilon u_{xx}(x,t) + \beta u_x(x,t) + \lambda(x)u(x,t) + g(x)U(x,t-D) \tag{5}$$

$$u(-\ell,t) = u(\ell,t) = 0 \tag{6}$$

$$u(x,0) = u_0(x) \tag{7}$$

where the state is defined in $(x,t) \in (-\ell,\ell) \times \mathbb{R}^+$, $\lambda(\cdot) \in C[-\ell,\ell]$, β, $\varepsilon > 0$ are known constants. Here, we assume that $g(\cdot) \in L^2([-\ell,\ell] : \mathbb{R}^+)$. With the change of variables $\breve{u}(x) = e^{\frac{\beta}{2\varepsilon}x}u(x)$, the system (5)–(7) is rewritten as

$$\breve{u}_t(x,t) = \varepsilon\breve{u}_{xx}(x,t) + \lambda_1(x)\breve{u}(x,t) + f(x)U(x,t-D) \tag{8}$$

$$\breve{u}(-\ell,t) = \breve{u}(\ell,t) = 0 \tag{9}$$

$$\breve{u}(x,0) = \breve{u}_0(t) \tag{10}$$

where $\lambda_1(x) = \lambda(x) - \frac{\beta^2}{4\varepsilon}$ and $f(x) = e^{\frac{\beta}{2\varepsilon}x}g(x)$. Following Krstic and Bresch-Pietri [33], the delayed input $U(x,t-D)$ can be written as a transport equation coupled with (8)–(10), considering the following infinite-dimensional actuator state:

$$v(x,s,t) = U(x,t+Ds-D). \tag{11}$$

Hence, system (8)–(10) is equivalent to

$$\breve{u}_t(x,t) = \varepsilon\breve{u}_{xx}(x,t) + \lambda_1(x)\breve{u}(x,t) + f(x)v(x,0,t) \tag{12}$$

$$\breve{u}(-\ell,t) = \breve{u}(\ell,t) = 0 \tag{13}$$

$$\breve{u}(x,0) = \breve{u}_0(t) \tag{14}$$

$$Dv_t(x,s,t) = v_s(x,s,t), \quad s \in [0,1] \tag{15}$$

$$v(x,1,t) = U(x,t) \tag{16}$$

$$v(x,s,0) = v_0(x,s) \tag{17}$$

where $v(x,s,t)$ is the state of the actuator and the known propagation speed is given by $1/D$. To design the delay-compensated controller $U(x,t)$, a backstepping approach can be employed [11], [12], [19], [27] by introducing the following integral transform:

$$z(x,s,t) = v(x,s,t) - \int_{-\ell}^{\ell}\gamma(x,s,y)\breve{u}(y,t)\mathrm{d}y - D\int_{-\ell}^{\ell}\int_0^s q(x,s,y,r)v(y,r,t)\mathrm{d}r\mathrm{d}y \tag{18}$$

where kernel function $\gamma(x,s,y)$ is defined on $[-\ell,\ell] \times [0,1] \times [-\ell,\ell]$ and $q(x,s,y,r)$ on $[-\ell,\ell] \times [0,1] \times [-\ell,\ell] \times [0,1]$. Using (18), the plant (12)–(17) is transformed into the following stable target system:

$$\breve{u}_t(x,t) = \varepsilon\breve{u}_{xx}(x,t) - c\breve{u}(x,t) + f(x)z(x,0,t) \tag{19}$$

$$\breve{u}(-\ell,t) = \breve{u}(\ell,t) = 0 \tag{20}$$

$$\breve{u}(x,0) = \breve{u}_0(x) \tag{21}$$

$$Dz_t(x,s,t) = z_s(x,s,t) \tag{22}$$

$$z(x,1,t) = 0 \tag{23}$$

$$z(x,s,0) = z_0(x,s) \tag{24}$$

where $c > 0$ is the controller gain, which potentially determines the convergence rate. The transport system in z has a mild solution

$$z(x,s,t) = \begin{cases} z_0(x,s+\frac{t}{D}), & 0 \le s+\frac{t}{D} \le 1 \\ 0, & s+\frac{t}{D} > 1 \end{cases}. \tag{25}$$

The gain kernels in (18) satisfy

$$\gamma_s(x,s,y) = D\varepsilon\gamma_{yy}(x,s,y) + D\lambda_1(y)\gamma(x,s,y) \tag{26}$$

$$\gamma(x,s,\ell) = \gamma(x,s,-\ell) = 0 \tag{27}$$

$$\gamma(x,0,y) = -\frac{\lambda_1(y)+c}{f(y)}\delta(x-y) \tag{28}$$

$$q_r(x,s,y,r) + q_s(x,s,y,r) = 0 \tag{29}$$

$$q(x,s,y,0) = f(y)\gamma(x,s,y) \tag{30}$$

where $\delta(\cdot)$ is Dirac delta function. From [34, Corollary 7.2.8], one know that (26)–(28) are well-posed and $q(x,s,y,r) = f(y)\gamma(x,s-r,y)$. From the boundary conditions (16) and (23), the associated control law is straightforwardly derived as

$$U(x,t) = \int_{-\ell}^{\ell} \gamma(x,1,y)\check{u}(y,t)\mathrm{d}y + \int_{-\ell}^{\ell}\int_{t-D}^{t} q\left(x,1,y,\frac{\kappa-t}{D}+1\right)U(y,\kappa)\mathrm{d}\kappa\mathrm{d}y. \quad (31)$$

Here, the stability of the target system (19)–(24) implies that of the original system (12)–(17), knowing that the transformation (18) is invertible with an inverse transform defined as

$$v(x,s,t) = z(x,s,t) + \int_{-\ell}^{\ell}\eta(x,s,y)\check{u}(y,t)\mathrm{d}y + D\int_{-\ell}^{\ell}\int_{0}^{s} p(x,s,y,r)z(y,r,t)\mathrm{d}r\mathrm{d}y. \quad (32)$$

The kernels function $\eta(x,s,y)$ and $p(x,s,y,r)$ are explicitly given as

$$\eta(x,s,y) = -\frac{\lambda_1(x)+c}{\ell f(x)}\sum_{n=1}^{\infty} e^{-D(c+\frac{n^2\pi^2\varepsilon}{\ell^2})s} \cdot \sin\left(\frac{n\pi}{\ell}y\right)\sin\left(\frac{n\pi}{\ell}x\right), \quad s\in(0,1] \quad (33)$$

$$\eta(x,0,y) = -\frac{\lambda_1(y)+c}{f(y)}\delta(x-y) \quad (34)$$

$$p(x,s,y,r) = f(y)\eta(x,s-r,y). \quad (35)$$

We refer the reader to the work in [12] and [19], where the proof of global stabilization of systems similar to (12)–(17) under the control law (31) is discussed. Next, we will derive an adaptive control law that globally stabilizes (12)–(17) when the input delay D is unknown.

III. Design of a Delay-Adaptive Feedback Control

A. Adaptive Controller Design

Considering the plant (8)–(10) with an unknown arbitrarily large delay D or equivalently the cascade system (12)–(17) with an unknown propagation speed $1/D$, our goal is to design an adaptive boundary controller that ensures a global stability result.

Assumption 1: The upper and lower bounds of the unknown $D>0$ denoted by $\bar{D}$ and $\underline{D}$, respectively, are known.

Based on the certainty equivalence principle, we define the following delay-adaptive distributed predictor feedback controller:

$$U(x,t) = \int_{-\ell}^{\ell}\gamma(x,1,y,\hat{D}(t))\check{u}(y,t)\mathrm{d}y + \int_{-\ell}^{\ell}\int_{t-\hat{D}(t)}^{t} q(x,1,y,\frac{\kappa-t}{\hat{D}(t)}+1,\hat{D}(t))U(y,\kappa)\mathrm{d}\kappa\mathrm{d}y \quad (36)$$

which is similar to (31), but account for the estimate of D, denoted $\hat{D}(t)$. The estimate $\hat{D}(t)$ is governed by an update law arising from the adaptive controller design, namely, $\dot{\hat{D}}(t)$.

B. Target System for the Plant With Unknown Input Delay

To prove the stability of the plant (8)–(10), equivalently, the system (12)–(17) under the control law (36), we introduce the backstepping transformation $(\check{u}, v)\mapsto(\check{u}, z)$ as

$$z(x,s,t) = v(x,s,t) - \int_{-\ell}^{\ell}\gamma(x,s,y,\hat{D}(t))\check{u}(y,t)\mathrm{d}y - \hat{D}(t)\int_{-\ell}^{\ell}\int_{0}^{s} q(x,s,y,r,\hat{D}(t))v(y,r,t)\mathrm{d}r\mathrm{d}y \quad (37)$$

and its inverse

$$v(x,s,t) = z(x,s,t) + \int_{-\ell}^{\ell}\eta(x,s,y,\hat{D}(t))\check{u}(y,t)\mathrm{d}y + \hat{D}(t)\int_{-\ell}^{\ell}\int_{0}^{s} p(x,s,y,r,\hat{D}(t))z(y,r,t)\mathrm{d}r\mathrm{d}y \quad (38)$$

where the kernels $\gamma(x,s,y,\hat{D}(t)), q(x,s,y,r,\hat{D}(t)), \eta(x,s,y,\hat{D}(t))$, and $p(x,s,y,r,\hat{D}(t))$ have the structure of $\gamma(x,s,y)$, $q(x,s,y,r)$, $\eta(x,s,y)$, and $p(x,s,y,r)$ defined in (26)–(30) and (33)–(35), respectively, but with D replaced by the estimate $\hat{D}(t)$. By using of (37), the system (12)–(17) is mapped into the following target system:

$$\check{u}_t(x,t) = \varepsilon\check{u}_{xx}(x,t) - c\check{u}(x,t) + f(x)z(x,0,t) \quad (39)$$

$$\check{u}(-\ell,t) = \check{u}(\ell,t) = 0 \quad (40)$$

$$\check{u}(x,0) = \check{u}_0(x) \quad (41)$$

$$Dz_t(x,s,t) = z_s(x,s,t) - \tilde{D}(t)P_1(x,s,t) - D\dot{\hat{D}}(t)P_2(x,s,t) \quad (42)$$

$$z(x,1,t) = 0 \quad (43)$$

$$z(x,s,t) = z_0(x,s) \quad (44)$$

where $\tilde{D}(t) = D - \hat{D}(t)$ is the estimation error, the functions $P_i(x,s,t), i=1,2$ are given as follows:

$$P_1(x,s,t) = \int_{-\ell}^{\ell} z(y,0,t)M_1(x,s,y,t)\mathrm{d}y + \int_{-\ell}^{\ell}\check{u}(y,t)M_2(x,s,y,t)\mathrm{d}y \quad (45)$$

$$P_2(x,s,t) = \int_{-\ell}^{\ell}\check{u}(y,t)M_3(x,s,y,t)\mathrm{d}y + \int_{-\ell}^{\ell}\int_{0}^{s} z(y,r,t)M_4(x,s,y,r,t)\mathrm{d}r\mathrm{d}y \quad (46)$$

and $M_i, i=1,2,3,4$ are defined as

$$M_1(x,s,y,t) = f(y)\gamma(x,s,y,\hat{D}(t)) \quad (47)$$

$$M_2(x,s,y,t) = \varepsilon\gamma_{yy}(x,s,y,\hat{D}(t)) - c\gamma(x,s,y,\hat{D}(t)) \quad (48)$$

$$M_3(x,s,y,t) = \gamma_{\hat{D}(t)}(x,s,y,\hat{D}(t)) + \hat{D}(t)\int_{-\ell}^{\ell}\int_{0}^{s} q_{\hat{D}(t)}(x,s,\xi,r,\hat{D}(t))\eta(\xi,r,y,\hat{D}(t))\mathrm{d}r\mathrm{d}\xi + \int_{-\ell}^{\ell}\int_{0}^{s} q(x,s,\xi,r,\hat{D}(t))\eta(\xi,r,y,\hat{D}(t))\mathrm{d}r\mathrm{d}\xi \quad (49)$$

$$M_4(x,s,y,r,t) = q(x,s,y,r,\hat{D}(t)) + \hat{D}q_{\hat{D}(t)}(x,s,y,r,\hat{D}(t)) + \hat{D}(t)\int_{-\ell}^{\ell}\int_{r}^{s} q(x,s,\xi,\delta,\hat{D}(t))p(\xi,\delta,y,r,\hat{D}(t))\mathrm{d}\delta\mathrm{d}\xi + \hat{D}(t)^2\int_{-\ell}^{\ell}\int_{r}^{s} q_{\hat{D}(t)}(x,s,\xi,\delta,\hat{D}(t)) \cdot p(\xi,\delta,y,r,\hat{D}(t))\mathrm{d}\delta\mathrm{d}\xi. \quad (50)$$

Here, $\bar{M}_i = \max_{-\ell\le x\le y\le \ell,\ 0\le s\le 1,\ t\ge 0}\{|M_i(x,s,y,t)|\}, i = 1,\ 2,3$, $\bar{M}_4 = \max_{-\ell\le x\le y\le\ell,\ 0\le r\le s\le 1,\ t\ge 0}\{|M_4(x,s,y,r,t)|\}$, and the time-dependence is induced by the estimate of the delay $\hat{D}(t)$.

C. Parameter's Update Law

To estimate the unknown parameter D, we choose the following update law:

$$\dot{\hat{D}}(t) = 2\theta \mathrm{b}_1 \mathrm{Proj}_{[\underline{D},\bar{D}]}\{\tau(t)\} \tag{51}$$

where $\theta \in (0, \theta^\star)$ and

$$\theta^\star = \frac{\min\{\underline{D}(c-\frac{2}{\iota_1}), \alpha, \underline{D}(\varepsilon + c)\}\cdot \min\{\frac{1}{2}, b_1\}}{4b_1^2 \bar{D} L^2} \tag{52}$$

$$\tau(t) = -\frac{\int_{-\ell}^{\ell}\int_0^1 (1+x)z(x,s,t)P_1(x,s,t)\mathrm{d}s\mathrm{d}x}{N(t)} \tag{53}$$

with

$$\alpha = b_1 - \frac{\bar{D}\bar{f}^2}{2}(\iota_1+\iota_2) \tag{54}$$

$$\begin{aligned} L = &\max\{2\ell\bar{M}_1, 2\ell\bar{M}_2, \bar{M}_1+\bar{M}_2, 1, 2\ell\bar{M}3\\ &\bar{M}_3 + \bar{M}_4 + 2\ell\bar{M}_4\}\end{aligned} \tag{55}$$

$$\begin{aligned} N(t) = &1 + \frac{1}{2}\|\check{u}\|^2 + \frac{1}{2}\|\check{u}_x\|^2\\ &+ b_1\int_{-\ell}^{\ell}\int_0^1 (1+s)z(x,s,t)^2\mathrm{d}s\mathrm{d}x \end{aligned} \tag{56}$$

$\bar{f} = \max_{-\ell\le x\le\ell}\{|f(x)|\}$, and b_1, ι_1, and ι_2 are positive constants, which will be discussed in Section IV. The standard projection operator introduced in (51) is given by

$$\mathrm{Proj}_{[\underline{D},\bar{D}]}\{\tau(t)\} = \begin{cases} 0 & \hat{D}(t) = \underline{D}\ \&\ \tau(t) < 0\\ 0 & \hat{D}(t) = \bar{D}\ \&\ \tau(t) > 0\\ \tau(t) & \text{otherwise}\end{cases}. \tag{57}$$

The convergence of the closed-loop system consisting of (12)–(17), with update law (51) and the adaptive controller (36), is stated in the following theorem.

Theorem 1: Consider the closed-loop system (12)–(17), the control law (36), and the update law defined through (51)–(57). The solution of the system $(\check{u}, v, D - \hat{D}(t))$ is stable and there exist positive constants R and ϱ (independent of the initial conditions) such that for all initial conditions satisfying $(u_0, v_0, \tilde{D}(0)) \in L^2(-\ell,\ell)\times L^2([-\ell,\ell]\times[0,1))\times[\underline{D},\bar{D}]$ and the compatibility conditions at the boundary, $u_0(-\ell) = u_0(\ell) = 0$ and $v(x,1,0) = U(x,0)$, the following holds:

$$\Psi(t) \le R(e^{\frac{\varrho}{\underline{D}}\Psi(0)} - 1)\forall t\ge 0 \tag{58}$$

where $\Psi(t) = \|\check{u}\|^2 + \|\check{u}_x\|^2 + \|v\|^2 + \tilde{D}(t)^2$, furthermore

$$\lim_{t\to\infty}\check{u}(x,t) = 0. \tag{59}$$

Remark 1: As stated in [33], the update law cannot guarantee the convergence of the estimated parameter to the real value, e.g., D, but one can obtain a perfect convergence when θ and b_1 are finely tuned. Nevertheless, the proposed adaptive control law guarantees that the closed-loop system is globally stable and the states converge to the desired set points, even if $\hat{D}(t)$ does not converge to D.

IV. Stability of the Closed-Loop System Under Delay-Adaptive Predictor Feedback Control

The proof of Theorem 1 is established by ensuring the pointwise boundedness and regulation of $\check{u}(x,t)$. As a first step, we state the L^2 boundedness of $\check{u}(x,t)$.

A. L^2 Boundedness of the Distributed State

We consider the Lyapunov–Krasovskii functional

$$\begin{aligned} V_1 = &D\log\Bigg(1 + \frac{1}{2}\|\check{u}\|^2 + \frac{1}{2}\|\check{u}_x\|^2\\ &+ b_1\int_{-\ell}^{\ell}\int_0^1(1+s)z(x,s,t)^2\mathrm{d}s\mathrm{d}x\Bigg) + \frac{\tilde{D}(t)^2}{2\theta}. \end{aligned} \tag{60}$$

Taking the time derivative of (60) along (39)–(44), we get

$$\begin{aligned} \dot{V}_1 = &\frac{1}{N(t)}\Bigg(\varepsilon D\int_{-\ell}^{\ell}\check{u}(x,t)\check{u}_{xx}(x,t)\mathrm{d}x - cD\int_{-\ell}^{\ell}\check{u}(x,t)^2\mathrm{d}x\\ &+ D\int_{-\ell}^{\ell}\check{u}(x,t)f(x)z(x,0,t)\mathrm{d}x - \varepsilon D\int_{-\ell}^{\ell}\check{u}_{xx}(x,t)^2\mathrm{d}x\\ &+ D\int_{-\ell}^{\ell}\check{u}_{xx}(x,t)(c\check{u}(x,t) - f(x)z(x,0,t))\mathrm{d}x\\ &+ 2b_1\int_{-\ell}^{\ell}\int_0^1(1+s)z(x,s,t)z_s(x,s,t)\mathrm{d}s\mathrm{d}x\\ &-2b_1\tilde{D}(t)\int_{-\ell}^{\ell}\int_0^1(1+s)z(x,s,t)P_1(x,s,t)\mathrm{d}s\mathrm{d}x\\ &-2b_1 D\dot{\hat{D}}(t)\int_{-\ell}^{\ell}\int_0^1(1+s)z(x,s,t)P_2(x,s,t)\mathrm{d}s\mathrm{d}x\Bigg)\\ &-\dot{\hat{D}}(t)\frac{\tilde{D}(t)}{\theta} \end{aligned} \tag{61}$$

where $N(t)$ is defined in (56). Then, by using integrations by part, Young's inequality, Poincaré inequality, and the update law (51), the following holds:

$$\begin{aligned} \dot{V}_1 \le &\frac{1}{N(t)}\Bigg(-\underline{D}\left(c - \frac{2}{\iota_1}\right)\|\check{u}\|^2 - \underline{D}(\varepsilon+c)\|\check{u}_x\|^2 - b_1\|z\|^2\\ &-\left(\varepsilon - \frac{1}{2\iota_2}\right)\|\check{u}_{xx}\|^2 - \left(b_1 - \bar{D}\left(\frac{\iota_1\bar{f}^2}{2} + \frac{\iota_2\bar{f}^2}{2}\right)\right)\|z(x,0,t)\|^2\\ &-2b_1 D\dot{\hat{D}}(t)\int_{-\ell}^{\ell}\int_0^1(1+s)z(x,s,t)P_2(x,s,t)\mathrm{d}s\mathrm{d}x\Bigg). \end{aligned} \tag{62}$$

Then, selecting $\iota_1 > \frac{2}{c}$, $\iota_2 \ge \frac{1}{2\varepsilon}$, and $b_1 > \frac{\bar{D}\bar{f}^2}{2}(\iota_1+\iota_2)$, we derive the following estimate:

$$\begin{aligned} \dot{V}_1 \le &\frac{1}{N(t)}\Bigg(-\underline{D}\left(c - \frac{2}{\iota_1}\right)\|\check{u}\|^2 - \underline{D}(\varepsilon+c)\|\check{u}_x\|^2 - b_1\|z\|^2\\ &- 2b_1 D\dot{\hat{D}}(t)\int_{-\ell}^{\ell}\int_0^1(1+s)z(x,s,t)P_2(x,s,t)\mathrm{d}s\mathrm{d}x\\ &-\left(b_1 - \frac{\bar{D}\bar{f}^2}{2}(\iota_1+\iota_2)\right)\|z(x,0,t)\|^2\Bigg). \end{aligned} \tag{63}$$

After a lengthy but simple calculation, using the Cauchy–Schwarz and Young's equalities, and combining (45) and (46), one can get the following estimates:

$$\int_{-\ell}^{\ell}\int_0^1 (1+s)z(x,s,t)P_1(x,s,t)\mathrm{d}s\mathrm{d}x \leq L(\|\check{u}\|^2+\|\check{u}_x\|^2+\|z\|^2+\|z(x,0,t)\|^2) \tag{64}$$

$$\int_{-\ell}^{\ell}\int_0^1 (1+s)z(x,s,t)P_2(x,s,t)\mathrm{d}s\mathrm{d}x \leq L\left(\|\check{u}\|^2+\|\check{u}_x\|^2+\|z\|^2\right) \tag{65}$$

where L is given by (55).

Thus, combining (63), (64), and (65), we have

$$\dot{V}_1 \leq -\left(\min\{\underline{D}(c-\frac{2}{\iota_1}),\alpha,\underline{D}(\varepsilon+c)\} - \theta\frac{4b_1^2\bar{D}L^2}{\min\{\frac{1}{2},b_1\}}\right) \cdot \frac{\|\check{u}\|^2+\|\check{u}_x\|^2+\|z\|^2+\|z(x,0,t)\|^2}{N(t)} \tag{66}$$

where α is defined in (54). By choosing $\theta \in (0,\theta^\star)$, where $\theta^\star$ is defined in (52), we have $\dot{V}_1 \leq 0$, and hence

$$V_1(t) \leq V_1(0) \tag{67}$$

for all $t \geq 0$. From this result, we now derive a stability estimate. The following inequalities readily follow from (60)–(67):

$$\|\check{u}(t)\|^2 \leq 2(e^{\frac{V_1}{\underline{D}}}-1), \quad \|\check{u}_x(t)\|^2 \leq 2(e^{\frac{V_1}{\underline{D}}}-1) \tag{68}$$

$$\|z(t)\|^2 \leq \frac{1}{b_1}(e^{\frac{V_1}{\underline{D}}}-1), \quad \tilde{D}(t)^2 \leq 2\theta V_1. \tag{69}$$

The relationship between $(\check{u},v)$ and $(\check{u},z)$ is given in the following proposition.

Proposition 1: From (37) and (38), we get the following relationship between the original and the target system:

$$\|\check{u}(t)\|^2+\|\check{u}_x(t)\|^2+\|v(t)\|^2 \leq s_1\|\check{u}(t)\|^2+s_2\|\check{u}_x(t)\|^2+s_3\|z(t)\|^2 \tag{70}$$

$$\|\check{u}(t)\|^2+\|\check{u}_x(t)\|^2+\|z(t)\|^2 \leq r_1\|\check{u}(t)\|^2+r_2\|\check{u}_x(t)\|^2+r_3\|v(t)\|^2 \tag{71}$$

where r_i and s_i, $i=1,2,3$ are sufficiently large positive constants given by

$$s_1 = 1+C_1, \quad s_2 = 1, \quad s_3 = 3+3\bar{D}^2C_2 \tag{72}$$

$$r_1 = 1+C_3, \quad r_2 = 1, \quad r_3 = 3+3\bar{D}^2C_4 \tag{73}$$

$C_i > 0$, for $i=1,2,3,4$ are constants.

The proof of Proposition 1 is stated in the Appendix. Furthermore, from (60), (68), (69), and (70), it follows that

$$\|\check{u}(t)\|^2+\|\check{u}_x(t)\|^2+\|v(t)\|^2 \leq \left(2s_1+2s_2+\frac{s_3}{b_1}\right)(e^{\frac{V_1}{\underline{D}}}-1) \tag{74}$$

and combining $\tilde{D}(t)^2 \leq 2\theta V_1$ and (74), we get

$$\Psi(t) \leq \left(2s_1+2s_2+\frac{s_3}{b_1}+2\underline{D}\theta\right)(e^{\frac{V_1}{\underline{D}}}-1). \tag{75}$$

So, we have bounded $\Psi(t)$ in terms of $V_1(t)$ and, thus, using (67), in terms of $V_1(0)$. Now, we have to bound $V_1(0)$ in terms of $\Psi(0)$. First, from (60), it follows that

$$V_1 \leq \frac{\bar{D}}{2}\|\check{u}(t)\|^2+\frac{\bar{D}}{2}\|\check{u}_x(t)\|^2+2b_1\|z(t)\|^2+\frac{\tilde{D}(t)^2}{2\theta} \leq \varrho\Psi(t) \tag{76}$$

where $\varrho = (\bar{D}+2b_1)(r_1+r_2+r_3)+\frac{1}{2\theta}$, and hence to $V_1(0) \leq \varrho\Psi(0)$. Then, combining (67) and (75), we have completed the proof of the stability estimate (58) with $R = 2s_1+2s_2+\frac{s_3}{b_1}+2\underline{D}\theta$.

B. Pointwise Boundedness and Regulation of the Distributed State

Now, we ensure the regulation of the distributed state. From (60), (66), and (67), we get $\|\check{u}\|$, $\|\check{u}_x\|$, $\|z\|$, and $\hat{D}(t)$ are bounded. The following estimate is established:

$$\int_0^t \|\check{u}(\tau)\|^2\mathrm{d}\tau \leq \sup_{0\leq\tau\leq t} N(\tau)\int_0^t \frac{\|\check{u}(\tau)\|^2}{N(\tau)}\mathrm{d}\tau \leq \frac{N(0)e^{\frac{\tilde{D}(0)^2}{2\theta}}(\log N(0)+\frac{\tilde{D}(0)^2}{2\theta})}{\min\{\underline{D}(c-\frac{2}{\iota_1}),\alpha,\underline{D}(\varepsilon+c)\}-\theta\frac{4b_1^2\bar{D}L^2}{\min\{\frac{1}{2},b_1\}}} \tag{77}$$

by using (67) and integrating (66) over $[0,t]$. Thus, we get $\|\check{u}\|$ is square integrable in time [35]. One can establish that $\|\check{u}_x\|$, $\|z\|$, and $\|z(x,0,t)\|$ are square integrable in time similarly. Since $\|\check{u}_x\|$ is bounded, using Agmon's inequality, the boundedness of $\check{u}(x,t)$ for all $x \in [-\ell,\ell]$ is ensured. By using (70), we have $\|v\|$ is bounded and square integrable in time. From (31), we get $\|U\|$ is bounded and integrable by using the Cauchy–Schwart inequality. Then, by using (11), we get the boundedness of $\|v(x,0,t)\|$ for $t \geq D$.

Next, we prove that $\frac{\mathrm{d}}{\mathrm{d}t}(\|\check{u}\|^2)$ is bounded by defining the following Lyapunov function:

$$V_2 = \frac{1}{2}\int_{-\ell}^{\ell}\check{u}(x,t)^2\mathrm{d}x. \tag{78}$$

It can be easily established that the time derivative of (78) satisfies the following estimate:

$$\dot{V}_2 = \int_{-\ell}^{\ell}\check{u}(x,t)u_t(x,t)\mathrm{d}x+\int_{-\ell}^{\ell}f(x)\check{u}(x,t)v(x,0,t)\mathrm{d}x \leq -\varepsilon\|\check{u}_x\|^2+\left(\bar{\lambda}_1+\frac{1}{2\iota_3}\right)\|\check{u}\|^2+\frac{\bar{f}^2\iota_3}{2}\|v(x,0,t)\|^2 \tag{79}$$

where $\iota_3 > 0$. Knowing that $\|\check{u}_x\|$, $\|\check{u}\|$, and $\|v(x,0,t)\|$ are bounded and square integrable in time, so we have $\dot{V}_2 \leq -2\varepsilon V_2 + f_2(t) < \infty$ with $f_2(t) = (\bar{\lambda}_1+\frac{1}{2\iota_3})\|\check{u}\|^2+\frac{\bar{f}^2\iota_3}{2}\|v(x,0,t)\|^2$, that means, $\frac{\mathrm{d}}{\mathrm{d}t}(\|\check{u}\|^2)$ is bounded for $t \geq D$. Since $\|\check{u}\|^2$ is bounded and integrable, by Barbalat's lemma (see [36, Lemma D.1]), we get $\|\check{u}\| \to 0$ as $t \to \infty$. By Agmon's inequality $\check{u}(x,t)^2 \leq 2\|\check{u}\|\|\check{u}_x\|$, which also leads to the regulation of $\check{u}(x,t)$ to zero uniformly in x and for $t \geq D$, i.e., (59). As so far, we have proved Theorem 1.

V. Application to a Surface Decontamination Model

To illustrate the feasibility of the proposed adaptive controller design, we consider the one-dimensional model of a decontamination process subject to an unknown input delay arising from the transport of the sprayed neutralizer from the mobile source to the infected surface, as shown in Fig. 1.

We consider a contaminated surface over which a decontamination product is deposited at a controlled rate by a moving disinfectant source $\mathcal{S}$ using a spraying mechanism. The density or concentration of the polluted surface and its surrounding environment are denoted $\rho(X,t)$ and ρ_∞, respectively. The source of the decontaminant sprayed on the surface is denoted $\mathcal{S}(X,t)$, and we assume the existence of a unknown delay D arising from the transport of the particles from the source location to the contaminated surface. The position of the moving sprayer along the spatial domain $[-\mathcal{L},\mathcal{L}]$ initialized at d_0 is described

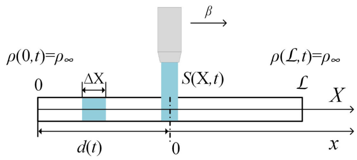

Fig. 1. One-dimensional domain of a spraying cleaning.

by the relative distance $d(t)$, namely, $d(t) = d_0 + \beta t$. The dynamics concentration of the pollutant obeys the following one-dimensional reaction–diffusion PDE:

$$\rho_t(X,t) = \varepsilon \rho_{XX}(X,t) + \varsigma(X)\rho(X,t) + \mathcal{S}(X,t-D)g(X-d(t)) \quad (80)$$

$$\rho(-\mathcal{L},t) = \rho(\mathcal{L},t) = \rho_\infty \quad (81)$$

where $X \in [-\mathcal{L},\mathcal{L}]$, $t > 0$, ε, $\mathcal{L} \in \mathbb{R}^+$, and $\varsigma(\cdot) \in C[-\mathcal{L},\mathcal{L}]$. The function g describes the distribution of the disinfectant controlled from the source $\mathcal{S}$, e.g., it works like a source power distribution, such as the Gaussian power distribution [37].

Remark 2: For this application, we choose Dirichlet boundary conditions assuming that the domain length is long enough to neglect change in concentration at the boundary, for Neumann or Robin boundary conditions, the control design is similar with that of the Dirichlet boundary conditions.

Introducing a coordinate system attached to the spray as $x(t) = X - d(t)$, we define the concentration distribution in the spray frame as : $\zeta(x,t) := \rho(x + d(t),t) = \rho(X,t)$ and $\lambda(x) := \varsigma(x + d(t)) = \varsigma(X)$. Note that following Zheng *et al.* [38], one can rewrite (80) and (81) as

$$\zeta_t(x,t) = \varepsilon\zeta_{xx}(x,t) + \beta\zeta_x(x,t) + \lambda(x)\zeta(x,t) + g(x)\mathcal{S}(x,t-D) \quad (82)$$

$$\zeta(-\ell,t) = \zeta(\ell,t) = \rho_\infty \quad (83)$$

where $x \in [-\ell,\ell], t > 0$ and $\ell \in \mathbb{R}^+$. For the desired power distribution, $\mathcal{S}^*(x)$, of the source decontaminant sprayed, we can solve for the desired density distribution $\zeta^*(y)$ by setting $\zeta_t(x,t) = 0$ in (82)

$$\varepsilon\zeta^*_{xx}(x) + \beta\zeta^*_x(x) + \lambda(x)\zeta^*(x) + g(x)\mathcal{S}^*(x) = 0 \quad (84)$$

$$\zeta^*(-\ell) = \zeta^*(\ell) = \rho_\infty. \quad (85)$$

Then, using $u(x,t) = \zeta(x,t) - \zeta^*(x)$, we get the following error system:

$$u_t(x,t) = \varepsilon u_{xx}(x,t) + \beta u_x(x,t) + \lambda(x)u(x,t) + g(x)\Delta\mathcal{S}(x,t-D) \quad (86)$$

$$u(-\ell,t) = u(\ell,t) = 0 \quad (87)$$

where $\Delta S(x,t-D) = S(x,t-D) - S^*(x)$. System (86), (87) is equivalent to (5), (6) with the initial data $u(x,0) = u_0(x)$.

In the simulation results, the real value of the delay is $D = 2$ assuming that the upper and lower bounds are $\bar{D} = 4$ and $\underline{D} = 1$, respectively. In the update law (51)–(57), we set $b_1 = 2$ and the adaptation gain $\theta = 0.0005$. The plant coefficients are chosen as $\ell = 2$, $\varepsilon = 10$, $\lambda = 6.5$, and $\beta = 2$. The spatial distribution of the sprayed neutralizer is defined as a Gaussian $g(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{x^2}{2\sigma^2}}$, where $\sigma = 0.7$. The simulation is performed considering the initial condition $\zeta_0(x) = 0.2\cos(\frac{\pi x}{2\ell}) + 1.5$, $x \in (-\ell,\ell)$ and the boundary conditions $\zeta_0(-\ell) = \zeta_0(\ell) = \rho_\infty = 0.5$. The steady-state boundary conditions [see (85)] are also set to $\zeta^* = 0.5$, which gives the desired power distribution $S^*(x) = -\frac{3.25}{g(x)}$.

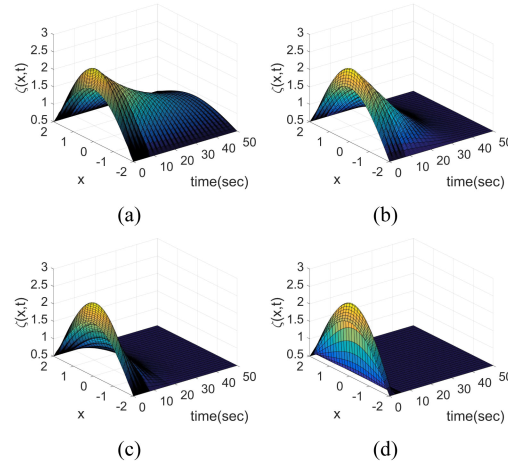

Fig. 2. Closed-loop system dynamics with $\zeta_0(x)$ and $\hat{D}(0)$. (a) Distributed state $\zeta(x,t)$ with nonadaptive control. (b) Distributed state $\zeta(x,t)$ with $\hat{D}(0) = 4$. (c) Distributed state $\zeta(x,t)$ with $\hat{D}(0) = 3$. (d) Distributed state $\zeta(x,t)$ with $\hat{D}(0) = 1$.

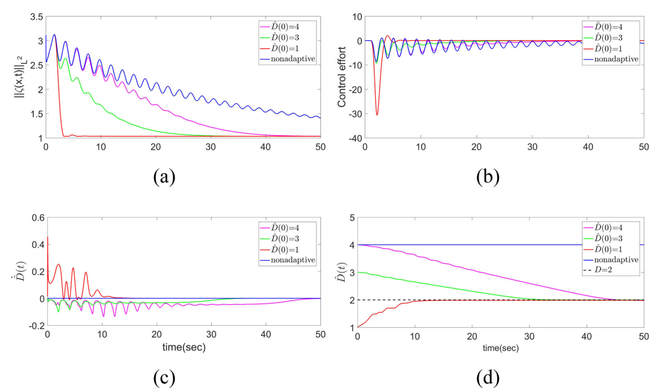

Fig. 3. Closed-loop system dynamics with $u_0(x)$ and to $\hat{D}(0)$ with and without adaptation. (a) L^2-norm of the distributed state $u(x,t)$. (b) Time evolution of the control signal. (c) Dynamics of the update law $\hat{D}(t)$. (d) Time evolution of the estimate of the unknown parameter $\hat{D}(t)$.

Fig. 2 shows the convergence of the distributed plant's state $\zeta(x,t)$ for four different cases: nonadaptive control with a mismatched delay $\hat{D}(t) = 4$, adaptive control with an initial delay guess $\hat{D}(0) = 4$, adaptive control with an initial delay guess $\hat{D}(0) = 3$, and adaptive control with an initial delay guess $\hat{D}(0) = 1$. Fig. 3(a) shows the dynamics of the L^2-norm of the distributed state with and without adaption. The adaptive controller enables faster convergence than the nonadaptive one for all initial conditions. The control effort is displayed as the average value for all x over time in Fig. 3(b) and the update law in Fig. 3(c). Finally, Fig. 3(d) reflects a good estimate of the delay $\hat{D}(t)$, which is converging to the true value $D = 2$.

VI. Conclusion

In this article, we design an adaptive controller and an unknown delay parameter's update law to stabilize a reaction–advection–diffusion system with unknown in domain input delay. Based on a Lyapunov argument, the adaptively controlled plant is globally stable in the L^2 norm. The theoretical statements are supported by considering a mobile robot that spread a neutralizer over a polluted surface to achieve efficient decontamination. Consistent simulation results to prove the effectiveness of the proposed method are provided. Further research will concern the design of an observer-based delay-adaptive boundary feedback control law for the same system.

Appendix A
Proof of Proposition 1

The proof of Proposition 1 is established using the following lemmas.

Lemma 1: Let $\eta(x,s,y,\hat{D})$ given as

$$\eta(x,s,y,\hat{D}) = -\frac{\lambda_1(x)+c}{\ell f(x)}\sum_{n=1}^{\infty} e^{-\hat{D}\left(c+\frac{n^2\pi^2}{\ell^2}\right)s} \cdot \sin\left(\frac{n\pi}{\ell}y\right)\sin\left(\frac{n\pi}{\ell}x\right) \tag{88}$$

and $F(\cdot)\in H_E^1[-\ell,\ell]$, $G(\cdot,\cdot)\in L_E^2([-\ell,\ell]\times[0,1])$, the following hold:

$$\int_{-\ell}^{\ell}\int_0^1\left|\int_{-\ell}^{\ell}\eta(x,s,y,\hat{D})F(y)\mathrm{d}y\right|^2\mathrm{d}s\mathrm{d}x \le C_1\|F\|^2 \tag{89}$$

$$\int_{-\ell}^{\ell}\int_0^1\left|\int_{-\ell}^{\ell}\int_0^s f(y)\eta(x,s-r,y,\hat{D})G(y,r)\mathrm{d}r\mathrm{d}y\right|^2\mathrm{d}s\mathrm{d}x \le C_2\|G\|^2. \tag{90}$$

Proof: Substituting (88) into the left of (89), it holds that

$$\int_{-\ell}^{\ell}\left|\int_{-\ell}^{\ell}\eta(x,s,y,\hat{D})F(y)\mathrm{d}y\right|^2\mathrm{d}x \le \frac{(\bar{\lambda}_1+c)^2}{\ell^2\underline{f}^2}\int_{-\ell}^{\ell}\left|\int_{-\ell}^{\ell}\sum_{n=1}^{\infty}e^{-\hat{D}(c+\frac{n^2\pi^2}{\ell^2})s} \cdot \sin\left(\frac{n\pi}{\ell}y\right)\sin\left(\frac{n\pi}{\ell}x\right)F(y)\mathrm{d}y\right|^2\mathrm{d}x \tag{91}$$

where $\underline{f} = \min_{-\ell\le x\le\ell}\{|f(x)|\}$. Using Parseval's theorem, we get

$$\int_{-\ell}^{\ell}\left|\int_{-\ell}^{\ell}\eta(x,s,y,\hat{D})F(y)\mathrm{d}y\right|^2\mathrm{d}x \le \frac{2\ell(\bar{\lambda}_1+c)^2}{\ell^2\underline{f}^2}\sum_{n=1}^{\infty}e^{-2\hat{D}(c+\frac{n^2\pi^2}{\ell^2})s}\left(\int_{-\ell}^{\ell}\sin(\frac{n\pi}{\ell}y)F(y)\mathrm{d}y\right)^2. \tag{92}$$

Hence, we have

$$\int_{-\ell}^{\ell}\int_0^1\left|\int_{-\ell}^{\ell}\eta(x,s,y,\hat{D})F(y)\mathrm{d}y\right|^2\mathrm{d}s\mathrm{d}x \le \frac{2(\bar{\lambda}_1+c)^2}{l\underline{f}^2}\sum_{n=1}^{\infty}\int_0^1 e^{-2\hat{D}(c+\frac{n^2\pi^2}{\ell^2})s}\mathrm{d}s \cdot\left(\int_{-\ell}^{\ell}\sin(\frac{n\pi}{\ell}y)F(y)\mathrm{d}y\right)^2 \le \frac{(\bar{\lambda}_1+c)^2}{2\underline{D}\underline{f}^2(c\ell^2+\pi^2)}\|F\|^2 \tag{93}$$

where we use Parseval's theorem again for $F(x)=\sum_{n=0}^{\infty}b_n\sin(n\pi x)$ and $b_n=\int_{-\ell}^{\ell}F(y)\sin(n\pi y)\mathrm{d}y$.

Similarly using Parseval's theorem, we have

$$\int_{-\ell}^{\ell}\int_0^1\left|\int_{-\ell}^{\ell}\int_0^s f(y)\eta(x,s-r,y,\hat{D})G(y,r)\mathrm{d}r\mathrm{d}y\right|^2\mathrm{d}s\mathrm{d}x \le \frac{(\bar{\lambda}_1+c)^2\bar{f}^2}{2\underline{D}\underline{f}^2(c\ell^2+\pi^2)}\|G\|^2 \tag{94}$$

as stated in (90). Thus, this lemma is completely proved.

Similarly, we state the following lemma that can be proved using the arguments stated earlier.

Lemma 2: Let $\gamma(x,s,y,\hat{D})$ is a solution of well-posed PDE (26)–(28), and $F(\cdot)\in H_E^1[-\ell,\ell]$, $G(\cdot,\cdot)\in L_E^2([-\ell,\ell]\times[0,1])$, the following hold:

$$\int_{-\ell}^{\ell}\int_0^1\left|\int_{-\ell}^{\ell}\gamma(x,s,y,\hat{D})F(y)\mathrm{d}y\right|^2\mathrm{d}s\mathrm{d}x \le C_3\|F\|^2 \tag{95}$$

$$\int_{-\ell}^{\ell}\int_0^1\left|\int_{-\ell}^{\ell}\int_0^s f(y)\gamma(x,s-r,y,\hat{D})G(y,r)\mathrm{d}r\mathrm{d}y\right|^2\mathrm{d}s\mathrm{d}x \le C_4\|G\|^2. \tag{96}$$

To prove Proposition 1, we derive the following estimate:

$$\int_{-\ell}^{\ell}\int_0^1 v(x,s,t)^2\mathrm{d}s\mathrm{d}x \le 3\int_{-\ell}^{\ell}\int_0^1 z(x,s,t)^2\mathrm{d}s\mathrm{d}x + 3\int_{-\ell}^{\ell}\int_0^1\left(\int_{-\ell}^{\ell}\eta(x,s,y,\hat{D})\check{u}(y,t)\mathrm{d}y\right)^2\mathrm{d}s\mathrm{d}x + 3\hat{D}^2\int_{-\ell}^{\ell}\int_0^1\left(\int_{-\ell}^{\ell}\int_0^s f(y)\eta(x,s-r,y,\hat{D}) \cdot z(y,r,t)\mathrm{d}r\mathrm{d}y\right)^2\mathrm{d}s\mathrm{d}x \le C_1\|\check{u}\|^2 + 3(1+\bar{D}^2C_2)\|z\|^2 \tag{97}$$

where we use Lemma 1 to obtain (70). Finally, using Lemma 2, one can establish (71), similarly.

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