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Port Hamiltonian formulation of a system of two conservation laws with a moving interface

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ABSTRACT

In this paper we consider the port Hamiltonian formulation of systems of two conservation laws defined on two complementary intervals of some interval of the real line and coupled by some moving interface. We recall first how two port Hamiltonian systems coupled by an interface may be expressed as an port Hamiltonian systems augmented with two variables being the characteristic functions of the two spatial domains. Then we consider the case of a moving interface and show that it may be expressed as the previous port Hamiltonian system augmented with an input, being the velocity of the interface and its conjugated output variable. We then discuss the interface relations defining the dynamics of the displacement of the interface and give an illustration with the simple example of two gases coupled by a moving piston.

1. Introduction

It has been shown that a large class of physical distributed parameter systems with boundary external variables admit a port Hamiltonian formulation called *boundary port Hamiltonian systems* [24,20,16]. This structure has led to various methods of analysis of the existence of solutions, their well-posedness and control in the linear case [17,15,25,27,26,14] but also for coupled distributed and localized parameter systems [22,18].

In this paper we shall investigate whether the port Hamiltonian formulation may be extended to systems of conservation laws coupled by some moving interface. Such systems occur in various cases when the system is heterogeneous in the considered spatial domain, leading to consider several phases. The most simple example (which we shall also consider here) consists in two fluids which are separated by some moving wall. This wall separates two phases, the two fluids which might have different properties and induces some discontinuities of some variables at the interface. These discontinuities are the consequence of the model of the interface defined by a set of interface relations. The wall separating the two fluids may permit, or not, a mass flow or a pressure discontinuity for instance. The interfaces arise in models of different chemical processes such as polymer nanoparticules in a fluid [13] or evaporation processes where an interface separates the domain of existence of liquid, vapor phase or their mixture [19]. Interfaces may also separate subdomains of the spatial domain, depending on the existence of constraints on some of the state variables such as the volume and leading to a change of causality of the dynamical model [9].

More precisely we are inspired by a classical approach developed for fixed interfaces, consisting in augmenting the system of conservation laws of the physical model with trivial conservation laws associated with the so-called *color functions* which are actually the characteristic functions of the spatial domains separated by the interface [12,11,3,5]. In a first instance we shall show that this system of conservation laws may be formulated as port Hamiltonian system with a pair of port variables associated with the interface. In a second instance we shall first generalize the previous approach to moving interfaces and show that it may again be formulated as a port Hamiltonian system by adding a second pair of port variables corresponding to the displacement of the interface. In this model, the velocity of the interface appears like an input and the interface relations defining the dynamics of the displacement of the interface are then defined as an external port-based model. In the whole paper the spatial domain is an interval of the real line and we shall consider systems of two conservation laws.

The sketch of the paper is the following. In a first part we consider two Hamiltonian systems of two conservation laws coupled by a fixed interface. We first recall the definition of Stokes-Dirac structures and the boundary port Hamiltonian formulation of a system of two conservation laws with flux variables deriving from a Hamiltonian. Then we recall the extended systems obtained by introducing color functions, associated with the characteristic functions of the

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spatial domains defined by the interface. We then define a Dirac structure and the port Hamiltonian formulation of the systems of conservation laws coupled by a fixed interface. In the second part we consider a moving interface and generalize the formulation of the coupled system of conservation laws before formulating it in the port Hamiltonian frame. Finally we introduce the interface relations defining the dynamics of the interface as a two-port element and give an illustration on the simple example of two gases coupled by a piston.

2. Two port Hamiltonian systems coupled by an interface

2.1. Port Hamiltonian system of two conservation laws

Let us recall briefly in this section the port Hamiltonian formulation of a system of two conservation laws according to [24], representing two physical domains in canonical interaction as they may arise in the description of the electrical transmission line, the vibrating string, the p-system [24,20,16].

We shall consider systems of two conservation laws

$$\partial_t x + \partial_z \mathcal{N}(x) = 0 \tag{2.1}$$

defined on some spatial domain being the interval $Z=[a,\ b]$ and with time $t \in \mathbb{R}^+$ with the 2-dimensional state vector $x(z,t) = \begin{pmatrix} x_1(z,t) \\ x_2(z,) \end{pmatrix}$. The flux variables are defined by

$$\mathcal{N}(x) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \delta_{x_1}\mathcal{H} \\ \delta_{x_2}\mathcal{H} \end{pmatrix} \tag{2.2}$$

generated by the Hamiltonian functional $\mathcal{H}(x) = \int_a^b H(x)\,dz$ with Hamiltonian density function $H(x)$ (where $\delta_x\mathcal{H}$ denotes the variational derivative of $\mathcal{H}$ with respect to x). Then the system of conservation laws (2.1) with the closure relations (2.2) may be rewritten as the Hamiltonian system

$$\partial_t x = \mathcal{J}\delta_x\mathcal{H} \tag{2.3}$$

generated by the Hamiltonian functional $\mathcal{H}(x)$ and defined with respect to the differential operator

$$\mathcal{J} = \begin{pmatrix} 0 & -\partial_z \\ \partial_z^* & 0 \end{pmatrix} \tag{2.4}$$

where ∂_z^* is the formal adjoint of the operator ∂_z. Indeed if the flux variables (2.2) satisfy the boundary conditions given by $\delta_{x_1}\mathcal{H}(a) = \delta_{x_1}\mathcal{H}(b) = \delta_{x_2}\mathcal{H}(a) = \delta_{x_2}\mathcal{H}(b) = 0$, then $\partial_z^* = -\partial_z$ and Eq. (2.3) is precisely (2.1) and (2.2). Under the same conditions, the operator $\mathcal{J}$ is skew-symmetric. Furthermore as it is a matrix differential operator with constant coefficients, it satisfies the Jacobi identities and is a Hamiltonian operator, defining a Poisson bracket on the functionals of the state variables [21].

However for control purposes, it should precisely be assumed that the variational derivatives (equal to the flux variables) do not vanish at the boundaries in order to allow for energy exchange of the system with its environment. Therefore the Hamiltonian system (2.3) is augmented with the boundary port variables

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \begin{pmatrix} \delta_{x_1}\mathcal{H} \\ \delta_{x_2}\mathcal{H} \end{pmatrix}\Big|_{a,b} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \delta_{x_1}\mathcal{H} \\ \delta_{x_2}\mathcal{H} \end{pmatrix}\Big|_{a,b} \tag{2.5}$$

and is thereby extended to a boundary port Hamiltonian system defined with respect to a Stokes-Dirac structure which extends the Hamiltonian operator (2.4) [24,20,16].

Let us recall the definition of a Dirac structure which will be extensively used in this paper.

Definition 1 (*Courant [8]*). Consider two real vector spaces, $\mathcal{F}$ the *space of flow variables* and $\mathcal{E}$ the *space of effort variables*, together with a pairing, that is, a bilinear product

$$\begin{aligned} &\mathcal{F} \times \mathcal{E} : \to \mathbb{R} \\ &(f,e) \mapsto \langle e, f \rangle \end{aligned} \tag{2.6}$$

which induces the symmetric bilinear form $\ll,\gg$ on the bond space $\mathcal{B} = \mathcal{F} \times \mathcal{E} \ni (f,e)$ of *conjugated power variables* defined as

$$\ll (f_1,e_1),(f_2,e_2) \gg := \langle e_1,\ f_2 \rangle + \langle e_2,\ f_1 \rangle, \quad (f_i,e_i) \in \mathcal{F} \times \mathcal{E} \tag{2.7}$$

A *Dirac structure* is a linear subspace $D \subset \mathcal{F} \times \mathcal{E}$ which is isotropic and co-isotropic that is satisfied, $\mathcal{D} = \mathcal{D}^\perp$, with $\perp$ denoting the orthogonal complement with respect to the bilinear form $\ll,\gg$.

Particular Dirac structures, called Stokes-Dirac structures, are associated with Hamiltonian differential operators [24,16,15]; here we recall the particular case of the Stokes-Dirac structure associated with the Hamiltonian operator $\mathcal{J}$ defined in (2.4).

Proposition 2 (*van der Schaft and Maschke [24]*). *The linear subspace of the bond space $\mathcal{B} = \mathcal{F} \times \mathcal{E}$, product space of the space of flow variables $\mathcal{F}$ and effort variables $\mathcal{E}$ where $\mathcal{F} = \mathcal{E} = L^2((a,b),\mathbb{R}^2) \times \mathbb{R}^2$ defined by*

$$\mathcal{D} = \left\{ \left(\begin{pmatrix} f_1 \\ f_2 \\ f_\partial \end{pmatrix}, \begin{pmatrix} e_1 \\ e_2 \\ e_\partial \end{pmatrix} \right) \in \mathcal{F} \times \mathcal{E} / \right. \\ \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \in H^1((a,b),\mathbb{R}^2)^2, \ \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \mathcal{J}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \\ \left. \text{and} \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix}\Big|_{a,b} \right\} \tag{2.8}$$

is a Dirac structure, called Stokes-Dirac structure, with respect to the pairing

$$\left\langle \begin{pmatrix} f_1 \\ f_2 \\ f_\partial \end{pmatrix}, \begin{pmatrix} e_1 \\ e_2 \\ e_\partial \end{pmatrix} \right\rangle = \int_a^b (f_1 e_1 + f_2 e_2)\,dz + e_\partial^\top \Sigma f_\partial$$

with

$$\Sigma = \text{diag}(-1,1) \tag{2.9}$$

In the same way as Hamiltonian systems are defined with respect to a Hamiltonian operator, boundary port Hamiltonian systems are defined with respect to Stokes-Dirac structures [23,10]. Again we refer to [24,16,15] for the general definition of boundary port Hamiltonian defined respect to Stokes-Dirac structure and will only recall the definition for the case of a system of two conservation laws.

Proposition 3 (*van der Schaft and Maschke [24]*). *The Hamiltonian system of two conservation laws (2.3) augmented with the port variables (2.5) is equivalent to*

$$\left(\begin{pmatrix} \partial_t x_1 \\ \partial_t x_2 \\ f_\partial \end{pmatrix}, \begin{pmatrix} \delta_{x_1}\mathcal{H} \\ \delta_{x_2}\mathcal{H} \\ e_\partial \end{pmatrix} \right) \in \mathcal{D} \tag{2.10}$$

and defines a boundary port Hamiltonian system.

As a consequence of the properties of the Stokes-Dirac structure [24], the Hamiltonian function satisfies the following balance equation:

$$\frac{d}{dt}H = -e_\partial^\top \Sigma f_\partial$$

2.2. Interconnection of port Hamiltonian systems through an interface

In this section we recall briefly how port Hamiltonian systems are coupled through their boundaries. Therefore consider two port Hamiltonian systems (2.10) which are defined in two spatial domains $[a, 0[$ and $]0, b]$ which are respectively two intervals of $\mathbb{R}^-$ and $\mathbb{R}^+$ and denote their state variables and Hamiltonian with exponent $-$ and $+$ depending on which half real line they are defined.

For boundary port Hamiltonian systems it is natural to express the interface relations using the boundary port variables (2.5) which are in fact, in the case of the canonical systems of two conservation laws, actually the flux variables (2.2) evaluated at the interface. In this case the spaces of flow and effort variables of the complete system are defined as the product spaces of the flow and effort spaces on each domain: $\mathcal{F} = \mathcal{E} = L^2((a,\ 0), \mathbb{R}^2) \times \mathbb{R}^2 \times L^2((0,\ b), \mathbb{R}^2) \times \mathbb{R}^2$ and the Dirac structure is defined with respect to the product operator of (2.4)

$$\begin{pmatrix} \mathcal{J} & 0_2 \\ 0_2 & \mathcal{J} \end{pmatrix}$$

Considering for instance the interface relations being the balance equation $\delta_{x_1^+}\mathcal{H}^+ + \delta_{x_1^-}\mathcal{H}^- = 0$ and the continuity equation $\delta_{x_2^+}\mathcal{H}^+ = \delta_{x_2^-}\mathcal{H}^-$, and writing them in vector notation, one obtains the linear relation between the conjugated power variables

$$\begin{pmatrix} \delta_{x_2^-}\mathcal{H}^- \\ \delta_{x_1^+}\mathcal{H}^+ \end{pmatrix}\Bigg|(0^+) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \delta_{x_1^-}\mathcal{H}^- \\ \delta_{x_2^+}\mathcal{H}^+ \end{pmatrix}\Bigg|(0^-) \tag{2.11}$$

they may immediately be interpreted as defining a Dirac structure on the boundary port variables at the interface. Then, by composition of Dirac structures, the two boundary port Hamiltonian systems are composed of a single boundary port Hamiltonian system with boundary port variables being defined by $\delta_{x^-}H^-(a)$ and $\delta_{x^+}H^+(b)$, according to (2.5) [24].

In the sequel of the paper, we shall consider the following interface relations where the pair of interface port variables (f_I, e_I) is introduced

$$f_I = \delta_{x_2^+}\mathcal{H}^+ = \delta_{x_2^-}\mathcal{H}^- \tag{2.12}$$

$$0 = \delta_{x_1^+}\mathcal{H}^+ + \delta_{x_1^-}\mathcal{H}^- + e_I \tag{2.13}$$

Eq. (2.12) is again a continuity equation and Eq. (2.13) is a balance equation with an external term e_I. These are commonly considered interface relations [12,5,3] consisting of the continuity equation of one of the flux variable (then called *privileged variable*) and the introduction of a source term at the interface, in the balance equation of the other flux variable [4]. Denoting $e_i^+ = \delta_{x_i^+}\mathcal{H}^+$ and $e_i^- = \delta_{x_i^-}\mathcal{H}^-$ with $i = 1, 2$, the interface relations (2.12) (2.13) define the linear relations between the conjugated power variables

$$\begin{pmatrix} e_2^- \\ e_1^+ \\ f_I \end{pmatrix}\Bigg|(0^+) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} e_1^- \\ e_2^+ \\ e_I \end{pmatrix}\Bigg|(0^-) \tag{2.14}$$

with respect to a nonlinear matrix and therefore define a Dirac structure.

The interface relations may of course be much more general using nonlinear functions of the flux variables at the interface, port variables coupled to a dynamical system: if the interface relations define a Dirac structure coupled to a dissipative port Hamiltonian systems then by composition of Dirac structures, a dissipative port Hamiltonian system is obtained on the product space of the state space of the subsystems [7] as for instance in [22,18].

However in the sequel we shall depart from this procedure of composition of boundary port Hamiltonian systems. Indeed, as a consequence of considering moving interfaces, time-varying spatial domains have to be considered. These do not appear explicitly as *variables* in the definition of boundary port Hamiltonian systems. This is the reason why, in the remaining of the paper, we shall use additional state variables, the characteristic functions of the time-varying spatial domains of each subsystem.

2.3. Augmenting the port Hamiltonian systems with color functions

2.3.1. Prolongation of the variables on the domain $[a,\ b]$

We shall follow the approach suggested in [12,5,3] where instead of considering the product spaces of the variables defined in the different spatial domains, the state variables of the coupled systems are defined on the composed spatial domain, the interval $[a,\ b]$. The interface at $z=0$ becomes then an interior point of the spatial domain, however some external variables are still associated with the interface. Following [1,6,2] we use the characteristic functions of the domains of the two systems

$$c_0(z,t) = \begin{cases} 1 & \forall z \in [a, 0[\\ 0 & \forall z \in [0, b] \end{cases} \quad \text{and} \quad \bar{c}_0(z,t) = \begin{cases} 1 & \forall z \in]0, b] \\ 0 & \forall z \in [a, 0] \end{cases} \tag{2.15}$$

Hence the state variables of the coupled system may be expressed as the sum of prolongations of the variables of each subsystem to the total spatial domain $Z = [a,\ b]$ by

$$x(z,t) = x^-(z,t) + x^+(z,t) \tag{2.16}$$

$$x^-(z,t) = c_0(z,t)x(z,t) \quad x^+(z,t) = \bar{c}_0(z,t)x(z,t) \tag{2.17}$$

And the flux variable of the two conservation laws becomes

$$\mathcal{N}(x,\ c_0,\ \bar{c}_0) = c_0\mathcal{N}^-(x) + \bar{c}_0\mathcal{N}^+(x) \tag{2.18}$$

with

$$c_0\mathcal{N}(x,\ c_0,\ \bar{c}_0) = c_0\ \mathcal{N}^-(x) \tag{2.19}$$

$$\bar{c}_0\mathcal{N}(x,\ c_0,\ \bar{c}_0) = \bar{c}_0\ \mathcal{N}^+(x) \tag{2.20}$$

where it should be noticed that $\mathcal{N}^-(x)$ and $\mathcal{N}^+(x)$ in (2.18), (2.19), (2.20) are different flux functions in general.

2.3.2. Conservation laws and interface relations as a single system of balance equations

We shall now consider the two systems of Hamiltonian conservation laws coupled by the interface relations defined in (2.12) and (2.13). As a consequence of these relations, considering the definition of the flux variables (2.2), it appears that the flux variable $\mathcal{N}_1$ satisfies a continuity equation at the interface whereas the flux variable $\mathcal{N}_2$ satisfies a balance equation at the interface.

In the first instance, let us consider the conservation law of the state variablex_1 which may be written as (on the whole domain $[a,\ b]$)

$$\begin{aligned} \partial_t x_1 &= -\partial_z(c_0\mathcal{N}_1^-(x) + \bar{c}_0\mathcal{N}_1^+(x)) \\ &= -\partial_z(c_0\mathcal{N}_1(x, c_0, \bar{c}_0) + \bar{c}\mathcal{N}_1(x, c_0, \bar{c}_0)) \\ &= -\underbrace{[\partial_z\ c_0 \cdot + \partial_z\ \bar{c}_0 \cdot]}_{\mathbf{d}_0}\mathcal{N}_1(x, c_0, \bar{c}_0) \end{aligned} \tag{2.21}$$

where the operator

$$\mathbf{d}_0 = -[\partial_z\ c_0 \cdot + \partial_z\ \bar{c}_0 \cdot] \tag{2.22}$$

acts as the differential operator $-\partial_z$ on each sub-domain (according to the system (2.3) and (2.4)).

Indeed (2.21) corresponds to the local formulation of the conservation laws on arbitrary domain $[a',\ b']$ with either on an interval $[a',\ b']$ on the negative real line $(a \le a' \le b' < 0\ )$

$$\frac{d}{dt}\int_{a'}^{b'} x_1(z,t) = -\mathcal{N}_1^-(a',t) + \mathcal{N}_1^-(b',t)$$

or on an interval $[a',\ b']$ on the positive real line $(0 < a' < b' \le b)$)

$$\frac{d}{dt}\int_{a'}^{b'} x_1(z,t) = -\mathcal{N}_1^+(a',t) + \mathcal{N}_1^+(b',t)$$

Now let us consider the formulation of the conservation law of x_1 on an arbitrary interval $[a',\ b']$ containing the interface (with $a \le a' < 0 < b' \le b$). The assumption the continuity of the flux variable $\mathcal{N}_1$ implies the following consequence on the conservation law of the variable x_1

$$\begin{aligned}\frac{d}{dt}\int_{a'}^{b'} x_1(z,t)\,dz &= \frac{d}{dt}\int_{a'}^{0} x_1(z,t)\,dz + \frac{d}{dt}\int_{0}^{b'} x_1(z,t)\,dz\\ &= \int_{a'}^{b'} \mathbf{d}_0\mathcal{N}_1(x, c_0, \overline{c}_0)\,dz\\ &= \int_{a'}^{b'} -[\partial_z\, c_0.+\partial_z\,\overline{c}_0.]\,\mathcal{N}_1(x,\ c_0,\ \overline{c}_0)\,dz\\ &= -\mathcal{N}_1^-(a',t)+\mathcal{N}_1^-(0^-,t)-\mathcal{N}_1^+(0^-,t)+\mathcal{N}_1^+(b',t)\\ &= -\mathcal{N}_1^-(a',t)+\mathcal{N}_1^+(b',t)\\ &= -\mathcal{N}_1(a',t)+\mathcal{N}_1(b',t)\\ &= -e_2(a',t)+e_2(b',t)\end{aligned} \tag{2.23}$$

In the second instance, let us consider the conservation law of the state variable x_2 and remind that, at the interface, the associated flux variable $\mathcal{N}_2$ is supposed to satisfy the balance equation (2.13) with the source term $e_I\delta(z)$ (a Dirac distribution), localized at the interface. But firstly we have to calculate the dual operator, denoted by $\mathbf{d}_0^*$, to the operator $\mathbf{d}_0$ defined in (2.22), in order to be able to express the power pairing. Therefore consider two effort variables e_1 and e_2 and compute

$$\begin{aligned}\int_a^b e_1(\mathbf{d}_0\ e_2)\,dz &= -\int_a^b (e_1[\partial_z\ c_0.+\partial_z\ \overline{c}_0.]e_2)\,dz\\ &= -\int_a^b (e_1[\partial_z(c_0e_2)+\partial_z(\overline{c}_0e_2)])\,dz\\ &= -[(c_0+\overline{c}_0)e_1e_2]_a^b + \int_a^b (c_0e_2+\overline{c}_0e_2)(\partial_z e_1)\,dz\\ &= -[(c_0+\overline{c}_0)e_1e_2]_a^b\\ &\quad + \int_a^b e_2[\partial_z\ c_0.+\ \partial_z\ \overline{c}_0.]e_1\,dz - \int_a^b e_2[(\partial_z\ c_0)+(\partial_z\ \overline{c}_0)]e_1\,dz\end{aligned}$$

Hence the dual operator is defined by

$$\begin{aligned}\mathbf{d}_0^* &= [\partial_z\ c_0.+\partial_z\ \overline{c}_0.]-[(\partial_z\ c_0)+(\partial_z\ \overline{c}_0)]\\ &= -\mathbf{d}_0+[(\partial_z\ c_0)+(\partial_z\ \overline{c}_0)]\end{aligned} \tag{2.24}$$

Using this dual operator the conservation law of the variable x_2 becomes

$$\partial_t x_2 = -\mathbf{d}_0^*\mathcal{N}_2 - e_I\delta(z) \tag{2.25}$$

where $\delta(z)$ denote the Dirac mass. Indeed, using similar calculation as in the preceding paragraph, one shows that (2.25) corresponds to the local formulation of the conservation laws on arbitrary interval $[a',\ b']$ on the negative real line $(a \le a' \le b' < 0\ )$ or on the positive real line $(0 < a' < b' \le b\ )$. On these intervals the operator $-\mathbf{d}_0^*$ acts as the differential operator $-\partial_z^*$ according to the Hamiltonian system (2.3) and (2.4).

On an arbitrary interval $[a',\ b']$ containing the interface (with $a \le a' < 0 < b' \le b$), the balance equation on the variable x_2 is

$$\begin{aligned}\frac{d}{dt}\int_{a'}^{b'} x_2(z,t) &= \int_{a'}^{b'} \{-\mathbf{d}_0^*\,\mathcal{N}_2(x,\ c_0,\ \overline{c}_0)-e_I(t)\delta(z)\}\,dz\\ &= \int_{a'}^{b'} \{(\mathbf{d}_0-[(\partial_z\ c_0)-(\partial_z\ \overline{c}_0)])\mathcal{N}_2(x,\ c_0,\ \overline{c}_0)-e_I(t)\delta(z)\}\,dz\\ &= \int_{a'}^{b'} \mathbf{d}_0\mathcal{N}_2(x,\ c_0,\ \overline{c}_0)\,dz\\ &\quad + \int_{a'}^{b'} \{([(\partial_z\ c_0)-(\partial_z\ \overline{c}_0)])\mathcal{N}_2(x,\ c_0,\ \overline{c}_0)-e_I(t)\delta(z)\}\,dz\\ &= -\mathcal{N}_2^-(a',t)+\mathcal{N}_2^-(0^-,\ t)-\mathcal{N}_2^+(0^-,\ t)+\mathcal{N}_2^+(b',t)\\ &\quad -\mathcal{N}_2^-(0^-,\ t)+\mathcal{N}_2^+(0^-,\ t)-\int_{a'}^{b'} e_I(t)\delta(z)\,dz\\ &= -\mathcal{N}_2^-(a',t)+\mathcal{N}_2^+(b',t)+e_I(t)\\ &= -e_1(a',t)+e_1(b',t)-e_I(t)\end{aligned} \tag{2.26}$$

On this balance equation, it appears clearly that the flux variables at the interface $\mathcal{N}_2^-(0^-,\ t)$ and $\mathcal{N}_2^+(0^-,\ t)$ are eliminated according to the balance equation (2.13).

2.3.3. *Hamiltonian system extended with color functions*

In the preceding paragraph we have formulated the dynamical equations off the system with an interface, as the system of balance equations (2.21) and (2.25) using the dual differential operators (2.22) and (2.24) which depend on the characteristic functions of the two domains separated by the interface. Following [12,5,3], we shall introduce explicitly these functions as variables of the system; they are then called *color functions* and will be denoted by $c(z,t)$ and $\overline{c}(z,t)$. Noticing that the spatial domains separated by the *fixed* interface are *constant*, hence also their characteristic functions c_0 and $\overline{c}_0$ defined in (2.15), it is clear that they satisfy the trivial conservation laws

$$\partial_t c = \partial_t \overline{c} = 0 \tag{2.27}$$

with initial conditions being precisely c_0 and $\overline{c}_0$ and compatible boundary conditions.

In the sequel we shall define an extended Hamiltonian system composed of the two balance equations (2.21) and (2.25) with closure equations (2.2) indexed by + and − for each spatial subdomain and augmented with the trivial conservation laws (2.27). Therefore define the Hamiltonian functional $\mathcal{H}(x,\ c,\ \overline{c}) = \int_a^b H(x,\ c,\ \overline{c})\,dz$ with density

$$H(x,\ c,\ \overline{c}) = c\ H^-(x) + \overline{c}\ H^+(x) \tag{2.28}$$

Denoting the *extended state variable* by

$$\tilde{x} = (x^T,\ c,\ \overline{c})^T \tag{2.29}$$

one computes the variational derivatives

$$\delta_{\tilde{x}}\mathcal{H}(\tilde{x}) = \begin{pmatrix}\delta_x\mathcal{H}(x,\ c,\ \overline{c})\\ \delta_c\mathcal{H}(x,\ c,\ \overline{c})\\ \delta_{\overline{c}}\mathcal{H}(x,\ c,\ \overline{c})\end{pmatrix} = \begin{pmatrix} c\delta_x\mathcal{H}^-(x)+\overline{c}\delta_x\mathcal{H}^+(x)\\ \mathcal{H}^-(x)\\ \mathcal{H}^+(x)\end{pmatrix} \tag{2.30}$$

Note that the first row corresponds precisely to the definition of the flux variable (2.18) for the particular solution c_0 and $\overline{c}_0$

$$\delta_x\mathcal{H}(x,\ c,\ \overline{c}) = c\delta_x\mathcal{H}^-(x)+\overline{c}\delta_x\mathcal{H}^+(x) = c\mathcal{N}^-(x)+\overline{c}\mathcal{N}^+(x) = \mathcal{N}(x,\ c,\ \overline{c})$$

This allows to augment the Hamiltonian system (2.3) with the trivial conservation laws of the color functions (2.27) obtaining the Hamiltonian system

$$\partial_t\tilde{x} = \mathcal{J}_a\delta_{\tilde{x}}\mathcal{H}(\tilde{x}) + Ie_I \tag{2.31}$$

$$I^T = (0\quad -1\quad 0\quad 0) \tag{2.32}$$

with respect to the operator

$$\mathcal{J}_a = \begin{pmatrix} 0 & \mathbf{d} & \\ -\mathbf{d}^* & 0 & 0_2\\ 0_2 & & 0_2\end{pmatrix} \tag{2.33}$$

where operator $\mathbf{d}$ is the nonlinear differential operator, modulated by $c(z,t)$ and $\bar{c}(z,t)$ defined by

$$\mathbf{d} = -[\partial_z\, c.+\partial\, \bar{c}.] \tag{2.34}$$

and its formal dual

$$\mathbf{d}^* = -\mathbf{d}+[(\partial_z\, c)-(\partial_z\, \bar{c})] \tag{2.35}$$

Furthermore the two operators satisfy, for any two effort variables e_1 and e_2 which do not vanish at the boundary

$$\int_a^b e_1(\mathbf{d}\; e_2)\, dz = \int_a^b e_2(\mathbf{d}^*\; e_1)\, dz - [(c+\bar{c})e_1 e_2]_a^b \tag{2.36}$$

2.3.4. Extension to a boundary Port Hamiltonian system

In order to take account of the energy exchange at the boundary $\{a,\ b\}$ and defining a conjugated flow variable f_I to the interface source term e_I at the interface, the Hamiltonian system (2.31) will now be extended to a port Hamiltonian systems with boundary and distributed ports. In the begin, the operator $\mathcal{J}_a$ defined in (2.33) and the input map at the interface defined by (2.32) are extended to a Stokes-Dirac structure using a similar procedure as in [24].

Proposition 4. *The set of relations $\mathcal{D}_I$ associated with a system of two conservation laws defined on the variables $\binom{x_1}{x_2}$ defined on a spatial domain $[a,\ b] \ni z$ with an interface at the point $z=0$ which imposes the continuity of the effort variable e_2 and allows for the discontinuity of the effort variable e_1 which is defined by*

$$\mathcal{D}_I = \left\{ \left(\begin{pmatrix} \tilde{f} \\ f_I \\ f_\partial \end{pmatrix}, \begin{pmatrix} \tilde{e} \\ e_I \\ e_\partial \end{pmatrix} \right) \in \mathcal{F}\times\mathcal{E} / \begin{pmatrix} \tilde{f} \\ f_I \end{pmatrix} = \begin{pmatrix} \mathcal{J}_a & I \\ -I^T & 0 \end{pmatrix} \begin{pmatrix} \tilde{e} \\ e_I \end{pmatrix} \text{ and } \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ (c+\bar{c}) & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}_{a,b} \right\} \tag{2.37}$$

with the flow variable $\tilde{f} = (f_1,\ f_2,\ f_c,\ f_{\bar{c}})^T$ and the effort variable $\tilde{e} = (e_1,\ e_2,\ e_c,\ e_{\bar{c}})^T$ associated with the extended state (2.29), the differential operator $\mathcal{J}_a$ defined in (2.33), the operators $\mathbf{d}$, resp. $\mathbf{d}^*$ defined in (2.34), resp. (2.35), the column vector I defined in (2.32), and bond space $\mathcal{B} = \mathcal{F}\times\mathcal{E}$ with $\mathcal{F} = \mathcal{E} = L^2((a,\ b),\mathbb{R})^5 \times \mathbb{R}^2$ endowed with the pairing

$$\left\langle \begin{pmatrix} \tilde{f} \\ f_I \\ f_\partial \end{pmatrix}, \begin{pmatrix} \tilde{e} \\ e_I \\ e_\partial \end{pmatrix} \right\rangle = \int_a^b \tilde{e}^T,\tilde{f}\; dz + e_\partial^T \Sigma f_\partial + \int_a^b e_I^T f_I\, dz \tag{2.38}$$

with Σ defined in (2.9) defines a Dirac structure.

One may notice immediately that the pair of port variables (e_I,f_I) at the interface is *distributed* variables. Let us prove in the sequel that the set (3.12) is indeed a Dirac structure. Note that we shall use the following notation:

$$\overline{f} = \begin{pmatrix} \tilde{f} \\ f_I \\ f_\partial \end{pmatrix}; \quad \overline{e} = \begin{pmatrix} \tilde{e} \\ e_I \\ e_\partial \end{pmatrix} \tag{2.39}$$

Let us first show the isotropy condition $\mathcal{D}_I \subset \mathcal{D}_I^\perp$

$$\langle\langle(\overline{f^1},\overline{e^1}),(\overline{f^2},\overline{e^2})\rangle\rangle = 0 \quad \forall(\overline{f^1},\overline{e^1}),(\overline{f^2},\overline{e^2}) \subset \mathcal{D}_I \tag{2.40}$$

with respect to the bilinear product associated with the pairing (2.38) and denoted in the sequel by $\mathcal{P}$

$$\begin{aligned} \mathcal{P} &= \langle\langle(\overline{f^1},\overline{e^1}),(\overline{f^2},\overline{e^2})\rangle\rangle \\ &= \langle\overline{f^1},\overline{e^2}\rangle + \langle\overline{f^2},\overline{e^1}\rangle \\ &= \int_a^b \tilde{e}^{2\mathrm{T}}\tilde{f}^1\; dz + \int_a^b \tilde{e}^{1\mathrm{T}}\tilde{f}^2\; dz + \int_a^b e_I^{1\mathrm{T}} f_I^2\; dz + \int_a^b e_I^{2\mathrm{T}} f_I^1\; dz \\ &\quad + e_\partial^{1\mathrm{T}}\Sigma^1 f_\partial^2 + e_\partial^{2T}\Sigma f_\partial^1 \end{aligned} \tag{2.41}$$

Using the constitutive relations of the set $\mathcal{D}_I$, the power product becomes

$$\begin{aligned} \mathcal{P} = &\int_a^b (e_1^2 \mathbf{d} e_2^1 + e_2^2[(-\mathbf{d}^*)e_1^1 - e_I^1])\; dz \\ &+ \int_a^b (e_1^1 \mathbf{d} e_2^2 + e_2^1[(-\mathbf{d}^*)e_1^2 - e_I^2])\; dz \\ &+ \int_a^b e_I^1 e_2^2\; dz + \int_a^b e_I^2 e_2^1\; dz + [(c+\bar{c})e_1^1 e_2^2]_a^b + [(c+\bar{c})e_1^2 e_2^1]_a^b \end{aligned}$$

and after noticing that the terms in the interface variables e_I^1 and e_I^2 cancel may be reorganized as follows:

$$\begin{aligned} \mathcal{P} = &\int_a^b (e_1^2 \mathbf{d} e_2^1 + e_2^1(-\mathbf{d}^*)e_1^2)\; dz + [(c+\bar{c})e_1^2 e_2^1]_a^b \\ &+ \int_a^b (e_2^2(-\mathbf{d}^*)e_1^1 + e_1^1 \mathbf{d} e_2^2)\; dz + [(c+\bar{c})e_1^1 e_2^2]_a^b \end{aligned}$$

and using the identity (2.36), one obtains that $\mathcal{P} = 0$ which proves the isotropy condition.

Let us now prove the co-isotropy condition $\mathcal{D}_I^\perp \subset \mathcal{D}_I$. This amounts to prove that if $(\overline{f}^2,\overline{e}^2) \in \mathcal{B}$ satisfies $\forall(\overline{f}^1,\overline{e}^1) \in \mathcal{D}_I$; $\langle\langle(\overline{f}^1,\overline{e}^1),\ (\overline{f}^2,\overline{e}^2)\rangle\rangle = 0$ then $(\overline{f}^2,\overline{e}^2) \in \mathcal{D}_I$. Therefore let us compute the bilinear product (2.41), assuming that $(\overline{f}^1,\overline{e}^1) \in \mathcal{D}_I$. One computes

$$\begin{aligned} \mathcal{P} = &\int_a^b (\tilde{e}^{2^\mathrm{T}}(\mathcal{J}_a \tilde{e}^1) + \tilde{e}^{1\mathrm{T}}\tilde{f}^2)\; dz + \int_a^b e_I^1 f_I^2\; dz + \int_a^b e_I^2 e_2^1\; dz \\ &+ ((c+\bar{c})e_1^1)|_{a,b}{}^{\mathrm{T}}\Sigma f_\partial^2 + e_\partial^{2\mathrm{T}}\Sigma(e_2^1)|_{a,b} \end{aligned} \tag{2.42}$$

Remind that, from the definition of $\mathcal{D}_I$, the variables $\tilde{e}^1$ and e_I^1 may be chosen freely.

Firstly, let us choose $e_1^1 = 0$, $e_2^1 = 0$, $e_I^1 = 0$ and $e_{\bar{c}}^1 = 0$. Then the bilinear product reduces to: $\mathcal{P} = \int_a^b e_c^1 f_c^2\; dz$ and the condition that it vanishes for any e_c^1 implies the relation $f_c^2 = 0$. By symmetry one obtains $f_{\bar{c}}^2 = 0$.

Secondly, let us choose $e_2^1 = 0$, $e_I^1 = 0$ and $e_1^1(a) = e_2^1(b) = 0$, then, using the definition of f_2 of the constitutive relations of $\mathcal{D}_I$ and (2.36) with zero boundary conditions, the bilinear product becomes

$$\begin{aligned} \mathcal{P} &= \int_a^b (e_2^2(-\mathbf{d}^* e_1^1) + e_1^1 f_1^2)\; dz \\ &= \int_a^b e_1^1(-\mathbf{d} e_2^2 + f_1^2)\; dz \end{aligned} \tag{2.43}$$

The condition that $\mathcal{P}$ vanishes for any e_1^1 hence implies that $f_1^2 = \mathbf{d} e_2^2$.

Thirdly, let us choose $e_1^1 = 0$, $e_I^1 = 0$ and $e_1^1(a) = e_2^1(b) = 0$, then, using the definition of f_1 of the constitutive relations of $\mathcal{D}_I$ and (2.36) with zero boundary conditions, the bilinear product becomes

$$\begin{aligned} \mathcal{P} &= \int_a^b (e_1^2(\mathbf{d} e_2^1) + e_2^1 f_2^2 + e_2^1 e_I^2)\; dz \\ &= \int_a^b e_2^1(\mathbf{d}^* e_1^2 + f_2^2 + e_I^2)\; dz \end{aligned} \tag{2.44}$$

The condition that $\mathcal{P}$ vanishes for any e_2^1 hence implies that $f_2^2 = -\mathbf{d}^* e_1^2 - e_I^2$.

Fourthly, let us choose $\tilde{e}_1^1 = 0$, $\tilde{e}_2^1 = 0$, then, using the definition of f_I of the constitutive relations of $\mathcal{D}_I$, the bilinear product becomes

$$\mathcal{P} = \int_a^b (-e_I^1 e_2^2 + f_I^2 e_I^1)\, dz$$
$$= \int_a^b e_I^1(-e_2^2 + f_I^2)\, dz \tag{2.45}$$

The condition that $\mathcal{P}$ vanishes for any e_I^1 hence implies that $f_I^2 = e_2^2$.

Fifth, let us choose $e_I^1 = 0$, then, using the constitutive relations of $\mathcal{D}_I$, the previously established relations on f_1^2, f_2^2 and $f_I^{\ 2}$, the relation (2.36), the bilinear product becomes

$$\mathcal{P} = \int_a^b (e_1^2 \mathbf{d} e_2^1 + e_2^2(-\mathbf{d}^*) e_1^1)\, dz + \int_a^b (e_1^1 \mathbf{d} e_2^2 + e_2^1(-\mathbf{d}^*) e_1^2)\, dz$$
$$+((c+\overline{c})e_1^1)|_{a,b}{}^{\mathrm{T}} \Sigma f_\partial^2 + e_\partial^{2\mathrm{T}} \Sigma (e_2^1)|_{a,b}$$
$$= -[(c+\overline{c})e_1^1 e_2^2]_a^b - [(c+\overline{c})e_1^2 e_2^1]_a^b$$
$$\times((c+\overline{c})e_1^1)|_{a,b}{}^{\mathrm{T}} \Sigma f_\partial^2 + e_\partial^{2\mathrm{T}} \Sigma (e_2^1)|_{a,b}$$

The condition that $\mathcal{P}$ vanishes for any $e_1^1(a)$ and $e_2^1(b)$ hence implies the boundary port variables and $\begin{pmatrix} f_\partial^2 \\ e_\partial^2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ (c+\overline{c}) & 0 \end{pmatrix} \begin{pmatrix} e_1^2 \\ e_2^2 \end{pmatrix}|_{a,b}$.

Comparing the constitutive relations of the Dirac structure $\mathcal{D}_I$ defined in (3.12) with the augmented Hamiltonian system (2.31), one may easily see that it may be endowed with a port Hamiltonian structure.

Corollary 5. *The augmented Hamiltonian systems (2.31) with the conjugated flow variable $f_I = e_2$ may be defined as a boundary port Hamiltonian system with respect to the Dirac structure $\mathcal{D}_I$ by*

$$\left(\begin{pmatrix} \partial_t \tilde{x} \\ f_I \\ f_\partial \end{pmatrix}, \begin{pmatrix} \delta_{\tilde{x}} H(\tilde{x}) \\ e_I \\ e_\partial \end{pmatrix} \right) \in \mathcal{D}_I$$

where the state vector $\tilde{x}$, defined in (2.29), the Hamiltonian $H(\tilde{x})$, defined in (2.28), the pair of port variables $(f_I,\ e_I)$ are associated with the interface and the pair of port variables $(f_\partial,\ e_\partial)$ is associated with the boundary of the spatial domain $[a,\ b]$.

As a consequence of the port Hamiltonian structure, the augmented Hamiltonian system (2.31) with the conjugated flow variable $f_I = e_2$ satisfies the following power balance equation:

$$\frac{d}{dt} H(x) = e_\partial^T \Sigma f_\partial + \int_a^b e_I^T f_I\, dz \tag{2.46}$$

Furthermore, if the Hamiltonians $H^+(x)$ and $H^+(x)$ are bounded from below,the augmented system has passivity properties. Indeed, although the Hamiltonian of the augmented system (2.31) is linear in the two color functions, they are invariants of the system hence, restricted to the invariant submanifold of invariance, it is indeed bounded from below.

Observe that the interface port variables $(f_I,\ e_I)$ are *distributed* variables on the complete domain $[a, b]$. If the color functions are the characteristic functions of the subdomains separated by the interface, then the power inflow at the interface appearing in the power balance equation (2.46) depends only on the values of the effort variables at the interface

$$\int_a^b e_I^T f_I\, dz = -e_1(0^-) e_2(0^+) + e_1(0^-) e_2(0^+)$$

involving the same variables as in (2.11).

3. Port Hamiltonian systems coupled through a moving interface

In this section we shall consider the case when the interface is *moving*. We shall denote by $l(t)$ the time-varying position of the interface in the interval $]a,\ b[$ and its velocity by $\dot{l}(t) = dl/dt$. In a first instance we shall show how the formulation as a port Hamiltonian systems of Corollary 5 may be extended to a moving interface. To this end *we shall consider the velocity $\dot{l}(t)$ of displacement of the interface as an input*. In the first instance we shall formulate the balance equations of the extended state variables $\tilde{x}$ defined in (2.29), for the case of a moving interface. In the second instance we shall show that they lead to a Port Hamiltonian system obtained by completing the system of Corollary 5 with an input relation and a conjugated port variable associated to $\dot{l}(t)$. We conclude with some remarks on the interface relations and treat the simple example of two gas in interaction through a piston.

3.1. Balance equations with moving interface

For a time-varying position $l(t)$ of the interface the spatial domains of the two subsystems are the intervals $[a, l(t)[$ and $]l(t), b]$. The two color functions, the characteristic functions of the domains, depend now on the position of the interface

$$c_{l(t)}(z,\ t) = \begin{cases} 1 & \forall z \in [a,\ l(t)[\\ 0 & \forall z \in [l(t),\ b] \end{cases} \tag{3.1}$$

and

$$\overline{c}_{l(t)}(z,\ t) = \begin{cases} 1 & \forall z \in]l(t),\ b] \\ 0 & \forall z \in [a,\ l(t)] \end{cases} \tag{3.2}$$

These color functions are the solutions of the transport equations depending on the velocity $\dot{l}(t)$ of the interface

$$\partial_t c(z,t) = -\dot{l}(t) \partial_z c(z,t) \quad \text{and} \quad \partial_t \overline{c}(z,t) = -\dot{l}(t) \partial_z \overline{c}(z,t) \tag{3.3}$$

with initial conditions

$$c(z,\ 0) = c_{l(0)}(z,\ t) \quad \text{and} \quad \overline{c}(z,\ 0) = \overline{c}_{l(0)}(z,\ t) \tag{3.4}$$

and compatible boundary conditions.

The state variables, the flux variables and the energy function may be defined according to the definitions (2.16), (2.18) and (2.28), respectively. We assume again that the interface relations (2.12) and (2.13) hold. Now due to the moving interface, the balance equations on intervals $[a',\ b']$ with $a \le a' < l(t) < b' \le b$ containing the interface will include an additional term, depending on the velocity $\dot{l}(t)$ of the interface.

The balance equation of the variable x_1 becomes

$$\frac{d}{dt} \int_{a'}^{b'} x_1(z,t)\, dz = \frac{d}{dt} \int_{a'}^{l(t)} x_1(z,t)\, dz + \frac{d}{dt} \int_{l(t)}^{b'} x_1(z,t)\, dz$$
$$= \int_{a'}^{l(t)} \partial_t x_1(z,t)\, dz + \int_{l(t)}^{b'} \partial_t x_1(z,t)\, dz$$
$$+\dot{l}(t)[x_1^-(l(t),t) - x_1^+(l(t),t)]$$
$$= \int_{a'}^{b'} \mathbf{d}_0 \mathcal{N}_1(x,\ c_l,\ \overline{c}_l)\, dz + \dot{l}(t)[x_1^-(l(t),t) - x_1^+(l(t),t)]$$
$$= -\mathcal{N}_1(a',t) + \mathcal{N}_1(b',t) + \dot{l}(t)[x_1^-(l(t),t) - x_1^+(l(t),t)]$$
$$= -e_2(a',t) + e_2(b',t) + \dot{l}(t)[x_1^-(l(t),t) - x_1^+(l(t),t)] \tag{3.5}$$

and its local formulation becomes

$$\partial_t x_1 = \mathbf{d}_0 \mathcal{N}_1(x,\ c_l,\ \overline{c}_l) + \dot{l}(t)[c\, x_1 \partial_z c + \overline{c}\, x_1 \partial_z \overline{c}] \tag{3.6}$$

For the state variable x_2 becomes, in a similar way as above

$$\frac{d}{dt}\int_{a'}^{b'} x_2(z,t) = \int_{a'}^{b'} \{-\mathbf{d}_0^* \mathcal{N}_2(x,\ c_l,\ \overline{c}_l) - e_l\}\, dz + \dot{l}(t)[x_1^-(l(t),t) - x_1^+(l(t),t)]$$
$$= \int_{a'}^{b'} \{(\mathbf{d}_0 - [(\partial_z\ c_l) - (\partial_z\ \overline{c}_l)])\mathcal{N}_2(x,\ c_l,\ \overline{c}_l) - e_l\}\, dz + \dot{l}(t)[x_1^-(l(t),t) - x_1^+(l(t),t)]$$
$$= -\mathcal{N}_2^-(a',t) + \mathcal{N}_2^+(b',t) + e_l(l(t)) + \dot{l}(t)[x_1^-(l(t),t) - x_1^+(l(t),t)]$$
$$= -e_1(a',t) + e_1(b',t) - e_l(l(t)) + \dot{l}(t)[x_2^-(l(t),t) - x_2^+(l(t),t)] \tag{3.7}$$

and its local formulation becomes

$$\partial_t x_1 = -\mathbf{d}_0^* \mathcal{N}_2(x,\ c_l,\ \overline{c}_l) - e_l + \dot{l}(t)[c\ x_1 \partial_z c + \overline{c}\ x_1 \partial_z \overline{c}] \tag{3.8}$$

3.2. Port Hamiltonian formulation

The four balance equations (3.3), (3.6) and (3.8) may be recognized as the augmented Hamiltonian formulation (2.31) of the system of two conservation laws with fixed interface which is completed with an additive term proportional the velocity. They may then be written in state space form

$$\partial_t \begin{pmatrix} x \\ c \\ \overline{c} \end{pmatrix} = \mathcal{J}_a \begin{pmatrix} \delta_x \mathcal{H}(x,\ c,\ \overline{c}) \\ \delta_c \mathcal{H}(x,\ c,\ \overline{c}) \\ \delta_{\overline{c}} \mathcal{H}(x,\ c,\ \overline{c}) \end{pmatrix} + I e_l + \dot{l}(t) \begin{pmatrix} cx & \overline{c}x \\ -1 & 0 \\ 0 & -1 \end{pmatrix} \partial_z \begin{pmatrix} c \\ \overline{c} \end{pmatrix} \tag{3.9}$$

with I defined in (2.32).

This defines an input map associated with the input $\dot{l}(t)$, velocity of the interface, as follows:

$$G(x,\ c,\ \overline{c}) = \begin{pmatrix} cx & \overline{c}x \\ -1 & 0 \\ 0 & -1 \end{pmatrix} \partial_z \begin{pmatrix} c \\ \overline{c} \end{pmatrix} \tag{3.10}$$

One may define then the *conjugated output* e_l by

$$e_l = \int_a^b \delta_{\tilde{x}} \mathcal{H}(\tilde{x})^T\ G(x,\ c,\ \overline{c})\, dz$$

which may also be defined as the pairing

$$e_l = \langle G(^{\mathrm{T}}x,\ c,\ \overline{c})|, \delta_{\tilde{x}} \mathcal{H}(\tilde{x})\rangle = \int_a^b \delta_{\tilde{x}} \mathcal{H}(\tilde{x})^T\ G(x,\ c,\ \overline{c})\, dz \tag{3.11}$$

This leads to define a Dirac structure associated for the system of conservation laws with a moving interface as follows.

Proposition 6. *The set of relations $\mathcal{D}_M$ associated with a system of two conservation laws defined with the variables $\binom{x_1}{x_2}$ on the spatial domain $[a,\ b] \ni z$ and an interface moving with velocity $\dot{l}$ which imposes the continuity of the effort variable e_2 and allows for the discontinuity of the effort variable e_1 which is defined by*

$$\mathcal{D}_M = \left\{ \left(\begin{pmatrix} \tilde{f} \\ f_I \\ e_l \\ f_\partial \end{pmatrix}, \begin{pmatrix} \tilde{e} \\ e_I \\ \dot{l} \\ e_\partial \end{pmatrix} \right) \in \mathcal{F} \times \mathcal{E} / \begin{pmatrix} \tilde{f} \\ f_I \\ -e_l \end{pmatrix} = \begin{pmatrix} \mathcal{J}_a & I & G(x,\ c,\ \overline{c}) \\ -I^T & 0 & 0 \\ -\langle G^T(x,\ c,\ \overline{c})| & 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{e} \\ e_I \\ \dot{l} \end{pmatrix} \right.$$
$$\left. \text{and} \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ (c+\overline{c}) & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}_{a,b} \right\} \tag{3.12}$$

with the flow variable $\tilde{f} = (f_1,\ f_2,\ f_c,\ f_{\overline{c}})^T$ and the effort variable $\tilde{e} = (e_1,\ e_2,\ e_c,\ e_{\overline{c}})^T$ associated with the extended state (2.29), the differential operator $\mathcal{J}_a$ defined in (2.33), the operators $\mathbf{d}$, resp. $\mathbf{d}^*$ defined in (2.34), resp. (2.35), the column vector I defined in (2.32), the input map G defined in (3.10) and its adjoint $\langle G^T|$ in (3.11) and bond space $\mathcal{B} = \mathcal{F} \times \mathcal{E}$ with $\mathcal{F} = \mathcal{E} = L^2((a,\ b), \mathbb{R})^5 \times \mathbb{R} \times \mathbb{R}^9$ endowed with the pairing

$$\left\langle \begin{pmatrix} \tilde{f} \\ f_I \\ e_l \\ f_\partial \end{pmatrix}, \begin{pmatrix} \tilde{e} \\ e_I \\ \dot{l} \\ e_\partial \end{pmatrix} \right\rangle = \int_a^b \tilde{e}^T \tilde{f}\, dz + \int_a^b {e_I}^{\mathrm{T}} f_I\, dz + e_\partial^T \Sigma f_\partial - e_l \dot{l} \tag{3.13}$$

with Σ defined in (2.9) defines a Dirac structure.

Proof. Let us first show the isotropy condition $\mathcal{D}_M \subset \mathcal{D}_M^\perp$.

Denote

$$\hat{f} = \begin{pmatrix} \tilde{f} \\ f_I \\ e_l \\ f_\partial \end{pmatrix}$$

and then pairing (3.13) by $\langle \hat{e},\ \hat{f} \rangle$. Then the isotropy condition is written as

$$\langle\langle (\hat{f}^1,\ \hat{e}^1), (\hat{f}^2,\ \hat{e}^2) \rangle\rangle = \langle \hat{e}^1,\ \hat{f}^2 \rangle + \langle \hat{e}^2,\ \hat{f}^1 \rangle = 0$$
$$\forall (\hat{f}^1,\ \hat{e}^1), (\hat{f}^2,\ \hat{e}^2)) \subset \mathcal{D}_M$$

with respect to the bilinear product associated with the pairing (3.13) or in an equivalent way

$$\langle \hat{e},\ \hat{f} \rangle = 0 \quad \forall (\hat{f},\ \hat{e}) \in \mathcal{D}_M$$

what is checked by computing the pairing

$$\langle \hat{e},\ \hat{f} \rangle = \int_a^b \tilde{e}^T \tilde{f}\ dz + \int_a^b {e_I}^{\mathrm{T}} f_I\, dz + e_\partial^T \Sigma f_\partial - e_l \dot{l}$$
$$= \int_a^b \tilde{e}^{\mathrm{T}} (\mathcal{J}_a \tilde{e} + I e_I + G\dot{l})\, dz + \int_a^b {e_I}^{\mathrm{T}} (-I^T \tilde{e})\, dz + {e_\partial}^{\mathrm{T}} \Sigma f_\partial - \left(\int_a^b \tilde{e}^T G\, dz \right) \dot{l}$$
$$= \left(\int_a^b {e_I}^{\mathrm{T}} I^T \tilde{e}\, dz + \int_a^b {e_I}^{\mathrm{T}} (-I^T \tilde{e})\, dz \right) + \left(\int_a^b \tilde{e}^{\mathrm{T}} \mathcal{J}_a \tilde{e}\, dz + {e_\partial}^{\mathrm{T}} \Sigma f_\partial \right) + \int_a^b \tilde{e}^{\mathrm{T}} G\dot{l}\, dz - \left(\int_a^b \tilde{e}^T G\, dz \right) \dot{l} = 0 \tag{3.14}$$

Let us now prove the co-isotropy condition $\mathcal{D}_M^\perp \subset \mathcal{D}_M$. This amounts to proving that if $(\hat{f}^2,\ \hat{e}^2) \in \mathcal{B}$ satisfies $\forall (\hat{f}^1,\ \hat{e}^1) \in \mathcal{D}_M$; $\langle\langle (\hat{f}^1,\ \hat{e}^1), (\hat{f}^2,\ \hat{e}^2) \rangle\rangle = 0$ then $(\hat{f}^2,\ \hat{e}^2) \in \mathcal{D}_M$. Therefore let us compute the bilinear product (3.14), assuming that $(\hat{f}^1,\ \hat{e}^1) \in \mathcal{D}_M$.

$$\mathcal{P}_e = \int_a^b (\tilde{e}^{2\mathrm{T}} (\mathcal{J}_a \tilde{e}^1 + I e_I^1 + G\dot{l}^1) + \tilde{e}^{1\mathrm{T}} \tilde{f}^2)\, dz + \int_a^b e_I^1 f_I^2\, dz + \int_a^b e_I^2 e_2^1\, dz \tag{3.15}$$

$$+ ((c+\overline{c}) e_1^1)|_{a,b}{}^{\mathrm{T}} \Sigma f_\partial^2 + e_\partial^{2\mathrm{T}} \Sigma (e_2^1)|_{a,b} \tag{3.16}$$

$$+\int_a^b [(\partial_z c)e_c^1 + (\partial_z \bar{c})e_{\bar{c}}^1]f_c^2\, dz + \int_a^b e_c^2(-\dot{l}^1)\, dz \tag{3.17}$$

$$-\left(\int_a^b G^T(x,\ c,\ \bar{c})\tilde{e}^1\, dz\right)\dot{l}^2 - e_l^2\dot{l}^1 \tag{3.18}$$

Remind that, from the definition of $\mathcal{D}_M$, the variables $\tilde{e}^1$, $e_l{}^1$ and $\dot{l}^1$ may be chosen freely.

In a first instance, choose $\dot{l}^1 = 0$. Then the power product becomes

$$\mathcal{P}_e = \int_a^b (\tilde{e}^{2^{\mathrm{T}}}(\mathcal{J}_a\tilde{e}^1 + \mathrm{I}e_l^1) + \tilde{e}^{1\mathrm{T}}\tilde{f}^2)\, dz + \int_a^b e_I^1 f_I^2\, dz + \int_a^b e_I^2 e_I^1\, dz \tag{3.19}$$

$$+((c+\bar{c})e_1^1)|_{a,b}{}^{\mathrm{T}}\Sigma f_\partial^2 + e_\partial^{2\mathrm{T}}\Sigma(e_2^1)|_{a,b} \tag{3.20}$$

$$+\int_a^b [(\partial_z c)e_c^1 + (\partial_z \bar{c})e_{\bar{c}}^1]f_c^2\, dz \tag{3.21}$$

$$-\left(\int_a^b G^T(x,\ c,\ \bar{c})\tilde{e}^1\, dz\right)\dot{l}^2 \tag{3.22}$$

Then, firstly, let us choose $e_1^1 = 0$, $e_2^1 = 0$, $e_l^1 = 0$ and $e_{\bar{c}}^1 = 0$. Then, the bilinear product reduces to

$$\mathcal{P}_e = \int_a^b e_c^1 f_c^2\, dz - \left(\int_a^b -\partial_z c\ e_c^1\, dz\right)\dot{l}^2 = \int_a^b (f_c^2 + \dot{l}^2\partial_z c)e_c^1\, dz$$

and the condition that it vanishes for any $e_c{}^1$ implies the relation $f_c^2 + \dot{l}^2\partial_z c = 0$.

By symmetry one obtains $f_{\bar{c}}^2 + \dot{l}^2\partial_z\bar{c} = 0$.

Secondly, let us choose $e_2^1 = 0$, $e_l^1 = 0$ and $e_1^1(a) = e_2^1(b) = 0$, then, using the preceding equalities, the definition of f_2 of the constitutive relations of $\mathcal{D}_M$ and (2.36) with zero boundary conditions, the bilinear product becomes

$$\mathcal{P}_e = \int_a^b (e_2^2(-\mathbf{d}^* e_1^1) + e_1^1 f_1^2)\, dz - \left(\int_a^b (cx_1\partial_z c + \bar{c}x_1\partial_z\bar{c})e_1^1\, dz\right)\dot{l}^2$$
$$= \int_a^b e_1^1(-\mathbf{d}e_2^2 + f_1^2 - (cx_1\partial_z c + \bar{c}x_1\partial_z\bar{c}))\, dz \tag{3.23}$$

The condition that $\mathcal{P}_e$ vanishes for any e_1^1 hence implies that $f_1^2 = \mathbf{d}e_2^2 + (cx_1\partial_z c + \bar{c}x_1\partial_z\bar{c})$.

Thirdly, let us choose $e_1^1 = 0$, $e_l^1 = 0$ and $e_1^1(a) = e_2^1(b) = 0$, then, using the preceding derived equalities and the definition of f_1 of the constitutive relations of $\mathcal{D}_M$ and (2.36) with zero boundary conditions, the bilinear product becomes

$$\mathcal{P}_e = \int_a^b (e_1^2(\mathbf{d}e_2^1) + e_2^1 f_2^2 + e_2^1 e_l^2)\, dz - \left(\int_a^b (cx_2\partial_z c + \bar{c}x_2\partial_z\bar{c})e_2^1\, dz\right)\dot{l}^2$$
$$= \int_a^b e_2^1(\mathbf{d}^* e_1^2 + f_2^2 + e_l^2 - (cx_2\partial_z c + \bar{c}x_2\partial_z\bar{c})\dot{l}^2)\, dz \tag{3.24}$$

The condition that $\mathcal{P}_e$ vanishes for any e_2^1 hence implies that $f_2^2 = -\mathbf{d}^* e_1^2 - e_l^2 + (cx_2\partial_z c + \bar{c}x_2\partial_z\bar{c})\dot{l}^2$.

Fourthly, let us choose $\tilde{e}_1^1 = 0, \tilde{e}_2^1 = 0$, then, using the definition of f_l of the constitutive relations of $\mathcal{D}_M$, the bilinear product becomes

$$\mathcal{P}_e = \int_a^b (-e_I^1 e_2^2 + f_I^2 e_I^1)\, dz$$
$$= \int_a^b e_I^1(-e_2^2 + f_I^2)\, dz \tag{3.25}$$

The condition that $\mathcal{P}_e$ vanishes for any $e_l{}^1$ hence implies that $f_I^2 = e_2^2$.

Fifth, let us choose $e_l^1 = 0$, then, using the constitutive relations of $\mathcal{D}_I$, the previously established relations on f_1^2, f_2^2 and $f_l{}^2$, the relation (2.36), the bilinear product becomes

$$\mathcal{P} = \int_a^b (e_1^2\mathbf{d}e_2^1 + e_2^2(-\mathbf{d}^*)e_1^1)\, dz + \int_a^b (e_1^1\mathbf{d}e_2^2 + e_2^1(-\mathbf{d}^*)e_1^2)\, dz$$
$$+((c+\bar{c})e_1^1)|_{a,b}{}^{\mathrm{T}}\Sigma f_\partial^2 + e_\partial^{2\mathrm{T}}\Sigma(e_2^1)|_{a,b}$$
$$= -[(c+\bar{c})e_1^1 e_2^2]_a^b - [(c+\bar{c})e_1^2 e_2^1]_a^b((c+\bar{c})e_1^1)|_{a,b}{}^{\mathrm{T}}\Sigma f_\partial^2 + e_\partial^{2\mathrm{T}}\Sigma(e_2^1)|_{a,b}$$

The condition that $\mathcal{P}_e$ vanishes for any $e_1^1(a)$ and $e_2^1(b)$ hence implies

$$\begin{pmatrix} f_\partial^2 \\ e_\partial^2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ (c+\bar{c}) & 0 \end{pmatrix}\begin{pmatrix} e_1^2 \\ e_2^2 \end{pmatrix}_{a,b}.$$

In a second instance, choose $\dot{l}^1 \neq 0$. Then, using the previously derived equalities, the power product becomes

$$\mathcal{P}_e = \int_a^b (\tilde{e}^{2T} G\dot{l}^1)\, dz + (-e_l^2\dot{l}^1)$$
$$= \left(\int_a^b \tilde{e}^{2T} G\, dz - e_l^2\right)\dot{l}^1 \tag{3.26}$$

The condition that $\mathcal{P}_e$ vanishes for any $\dot{l}^1$ hence implies $e_l^2 = \int_a^b G^T\tilde{e}^2\, dz$.

The port-Hamiltonian formulation of the system of two conservation laws with a moving interface with velocity $\dot{l}$ may be formulated as a port-Hamiltonian system.

Corollary 7. *The augmented Hamiltonian systems (3.9) with the conjugated interface flow variable $f_I = e_2$ and conjugated variable e_I to the interface velocity, defined in (3.11), may be defined as a boundary port-Hamiltonian system with respect to the Dirac structure $\mathcal{D}_M$ by*

$$\left(\begin{pmatrix} \partial_t\tilde{x} \\ f_I \\ e_l \\ f_\partial \end{pmatrix}, \begin{pmatrix} \delta_{\tilde{x}}\mathcal{H} \\ e_I \\ \dot{l} \\ e_\partial \end{pmatrix}\right) \in \mathcal{D}_M \tag{3.27}$$

where the state vector $\tilde{x}$, defined in (2.29), the Hamiltonian $H(\tilde{x})$, defined in (2.28), the pair of port variables $(f_I,\ e_I)$ at the interface, the pair of port variables $(\dot{l},\ e_l)$ are associated with the velocity of the interface and the pair of port variables $(f_\partial,\ e_\partial)$ is associated with the boundary of the spatial domain $[a,\ b]$.

Computing the balance equation for the Hamiltonian we find

$$\frac{d}{dt}H(x) = e_\partial\Sigma f_\partial + \int_a^b e_I^T f_I\, dz + \dot{l}e_l \tag{3.28}$$

It may be observed that, when restricting to the particular solution of the color functions (3.2) obtained with initial conditions (3.4), one can relate these port variables to interfacial effort and flux variables in a similar way as for a fixed interface. In this case the balance equation of the Hamiltonian is given by

$$\frac{d\mathcal{H}}{dt} = e_\partial\Sigma f_\partial + e_1(l^-)e_2(l^+) - e_1(l^-)e_2(l^+) - \dot{l}e_l$$

with output conjugated to the velocity of the interface being the *discontinuity of energy density at the interface*

$$e_l = (-\mathcal{H}^-(l) + \mathcal{H}^+(l))$$

3.3. Model of the interface's displacement

In the preceding section we have defined the dynamic model of a system of two conservation laws coupled by a moving interface with velocity $\dot{l}$ considered as an input variable and the interface relations (2.12) and (2.13). The port Hamiltonian model with the moving interface admits as port variables, the port variables (f_I, e_I) associated with the flux variables at the interface and the port variables $(\dot{l}, e_l)$ associated with the displacement of the interface. In this section we shall discuss possible closure relations which could be imposed of these two pairs of port variables at the interface and illustrate it on a very simple example: two gases with a piston at the interface.

In a first instance one should observe that the dynamics of displacement of the interface is necessarily finite-dimensional while the port variables (f_I, e_I) are distributed. Coming back to the motivating example of a thin interface, that is located at some point $l(t)$ which was the departure for the definition of the model in the Section 3.1, the port variables (f_I, e_I) may be related to a finite-dimensional pair of variables $(\phi_I, \varepsilon_I) \in \mathbb{R}^2$ with the following adjoint relations:

$$\begin{pmatrix} \phi_I \\ e_I \end{pmatrix} = \begin{pmatrix} \int_a^b \delta(z-l) f_I \, dz \\ \varepsilon_I \delta(z-l) \end{pmatrix} \tag{3.29}$$

which preserves the power product $\phi_I \varepsilon_I = \int_a^b e_I f_I \, dz$.

It should be noted that one could also define a thick interface by choosing another kernel than $\delta(z-l)$, with positive values and finite support.

In a second instance, one has to complete the interface relation with the dynamics of the position of the interface $l(t)$ for instance in terms of a port Hamiltonian system with state variables including $l(t)$ and the port variables (ϕ_I, ε_I) and $(\dot{l}, e_l)$. In this case by interconnection of port Hamiltonian systems through a Dirac structure one may conclude that the complete system is again port Hamiltonian and use its properties for the proof of well-posedness and passivity-based control design.

Example 8. Let us conclude this paragraph with *the example of two isentropic gases* (modeled by a systems of two boundary port Hamiltonian systems [24]) *coupled at their interface by some piston in motion*. The port Hamiltonian model of the gases is given precisely by Proposition 3 with state variables being the specific volume $x_1(t,z) = \mathfrak{v}(t,z)$, the velocity $x_2(t,z) = v(t,z)$, Hamiltonian is the sum of the internal energy density $\mathcal{U}(\mathfrak{v})$ and the kinetic energy density $\mathcal{H}(\mathfrak{v}, v) = \mathcal{U}(\mathfrak{v}) + v^2/2$. The variational derivative of the Hamiltonian is then

$$\begin{pmatrix} \delta_{\mathfrak{v}} H \\ \delta_v H \end{pmatrix} = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} -p(\mathfrak{v}) \\ v \end{pmatrix}$$

where $p(\mathfrak{v}) = -\delta_{\mathfrak{v}} U(\mathfrak{v})$ is the pressure. The interface relation (2.12) corresponds to the continuity of the effort variable $e_2 = v$ at the interface, which is the usual hypothesis that there is no cavitation at the piston and that the velocities of the fluids on both sides of the piston are equal to the velocity of the piston. And the interface relation (2.13) corresponds to the balance of forces exerted on the piston by the pressures $e_1 = -p(\mathfrak{v})$ of the gases from both gases and the external force f_I.

The system of the two gases with a moving interface is then formulated by Corollary 7 with the color functions being the characteristic functions of each subdomain. As the piston is considered as a thin interface, we use the relation (3.29). In order to complete the interface relations we shall assume that the piston has no mass but is subject to friction with coefficient ν and an linear elastic force with stiffness k. In this case the dynamics of the piston is defined as a simple integrator

$$\frac{dl}{dt} = \phi_I = v$$

and the conjugated effort variable is the sum of all forces applying on the piston

$$\varepsilon_i = -kl - \nu \phi_I$$

It may be interpreted as a finite-dimensional port Hamiltonian system with state variable l, structure matrix being zero, Hamiltonian function $\frac{1}{2} k l^2$ port-variables (ϕ_I, ε_I) and dissipative term. Finally the model has to be coupled with the pair of port variables $(\dot{l}, e_l)$. One relation is trivial

$$\dot{l} = \phi_I = v$$

The second one is less trivial and involves the effort variable e_l which is, when the color functions are the characteristic functions of both subdomains, the difference of the Hamiltonian density function at the interface $e_l = (-\mathcal{H}^-(l) + \mathcal{H}^+(l))$. The most simple way of defining some relation is to impose the continuity of the Hamiltonian density (which plays then the role of a *privileged variable*) which indeed completes the boundary conditions at the interface

$$e_l = 0$$

As a consequence using the total energy of the conservation laws and the interface model $\mathrm{H}_{tot}(\mathfrak{v}, v, l) = \int_a^b (\mathcal{U}(\mathfrak{v}) + v^2/2) \, dz + \frac{1}{2} k l^2$ one obtains the power balance equation

$$\frac{d\mathrm{H}_{tot}}{dt} = -\nu v^2 - v^-(a) p^-(\mathfrak{v})(a) + v^+(b) p^+(\mathfrak{v})(b)$$

4. Conclusion

In this paper we have suggested port Hamiltonian formulation of a system of two conservation laws (on a 1-dimensional spatial domain) coupled by a moving interface. We have firstly augmented the system of conservation laws with two transport equations of the characteristic functions of the subdomains defined by the interface. Then we have derived the port Hamiltonian formulation of this augmented system with, in addition to the boundary port variables at the boundary of the total domain, two pairs of port-variables associated with the interface. The first pair corresponds to a particular choice of interface relation corresponding to a continuity and a balance equation on the flux variables at the interface and the second pair is defined by the velocity of interface and its conjugated variable. Finally we have illustrated this model with the example of two gases coupled by a moving piston.

This is the first step towards considering the coupling through an interface of Hamiltonian systems composed of an arbitrary number of conservation laws. However the most interesting feature of this formulation is that it makes explicit the pairs of conjugated variables needed to express the interface relations when derived from a port Hamiltonian formulation. This might be a powerful insight in the various suggested interface relations in the literature and toward a passivity-based definition and classification of these interface relations.

Finally this port Hamiltonian formulation might open the way to the analysis of the well-posedness of these systems (in the continuation of [15,27]) as well as their passivity-based control which will be the aim of future work.

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