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Global exponential stability of event-triggered control of a parabolic PDE with switching dynamic triggering designs

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This paper presents dynamic design techniques—namely, continuous-time event-triggered control (CETC), periodic event-triggered control (PETC) and self-triggered control (STC)—for a class of unstable one-dimensional reaction-diffusion partial differential equations (PDEs) with boundary control and an anti-collocated sensing mechanism. For the first time, global exponential stability (GES) of the closed-loop system is established using a PDE backstepping control design combined with dynamic event-triggered mechanisms for parabolic PDEs—a result not previously achieved even under full-state measurement. When the emulated continuous-time backstepping controller is implemented on the plant using a zero-order hold, our design guarantees L^2-GES through the integration of novel switching dynamic event triggers and a newly developed Lyapunov functional. While CETC requires the continuous monitoring of the triggering function to detect events, PETC only requires the periodic evaluation of this function. The STC design assumes full-state measurements and, unlike CETC, does not require continuous monitoring of any triggering function. Instead, it computes the next event time at the current event time using only full-state measurements available at the current event time and the immediate previous event time. Thus, STC operates entirely with event-triggered measurements, in contrast to CETC and PETC, which rely on continuous measurements. The well-posedness of the closed-loop systems under all three strategies is established, and simulation results are provided to illustrate the theoretical results.

Keywords: PDE backstepping; event-triggered control; periodic event-triggered control; self-triggered control; global exponential stability; dynamic event-triggering.

1. Introduction

Event-triggered control (ETC) is an advanced feedback control strategy that updates the control signal based on specific triggering conditions rather than at fixed time intervals, as seen in traditional periodic sampled-data control. By selectively executing control updates only when necessary, ETC minimizes bandwidth usage, reduces computational demands and conserves energy, making it particularly advantageous for networked control systems (NCS) and embedded applications.

An ETC system has two integrated components: a feedback control law that ensures stability and a triggering mechanism that determines when control updates take place. For ETC to function properly and avoid Zeno behaviour—where infinite updates occur in finite time—a minimum time interval must be enforced between consecutive events. Consequently, the primary challenge in designing a triggering condition lies in ensuring the existence of a lower bound for the minimum dwell time (MDT). Static triggering mechanisms use fixed conditions based on the system's state and output to decide when an update is necessary. These conditions are predefined and typically involve threshold-based rules, making implementation straightforward but potentially conservative in terms of triggering frequency. Dynamic triggering mechanisms, on the other hand, introduce additional dynamics, such as internal variables or timers, to adaptively regulate the triggering condition. This flexibility helps reduce unnecessary updates while still maintaining stability and performance.

ETC for finite-dimensional systems. Building on early research in digital computer design for closed-loop systems (Monaco & Normand-Cyrot, 1985; Hsu & Sastry, 1987), foundational contributions to ETC include the development of event-based proportional–integral–derivative controller (PID) control (Åarzén, 1999) and event-based sampling for first-order stochastic systems (Åström & Bernhardsson, 1999). Over the past decade, the field has grown significantly, with novel design strategies emerging for systems governed by ordinary differential equations (ODEs) (Mazo & Tabuada, 2008; Heemels *et al.*, 2012; Girard, 2015; Liu & Jiang, 2015; Tallapragada & Cortés, 2016; Taylor *et al.*, 2021). This progress has further extended to ETC frameworks for systems described by partial differential equations (PDEs), broadening the scope of applications and theoretical advancements in the field.

ETC for infinite-dimensional systems. Both parabolic and hyperbolic class of PDEs have been studied through the lens of ETC:

- Recent progress in ETC for parabolic PDEs addressing both static and dynamic event-triggering mechanisms includes Espitia *et al.* (2021); Katz *et al.* (2021); Rathnayake & Diagne (2022); Rathnayake *et al.* (2022a,b); Kang *et al.* (2023); Koga *et al.* (2023); Wang & Krstic (2023a); Demir *et al.* (2024); Koudohode *et al.* (2024); Lhachemi (2024) and Rathnayake & Diagne (2024a). Full-state feedback static event-triggering mechanisms for reaction-diffusion (RD) PDEs using PDE backstepping are proposed in Espitia *et al.* (2021) and Koudohode *et al.* (2024), while dynamic event-triggering mechanisms are found in Rathnayake *et al.* (2022a,b) and Wang & Krstic (2023a). Adaptive dynamic ETC is presented in Wang & Krstic (2023a), where uncertain plant parameters are estimated through update laws. Observer-based modal decomposition methods for dynamic ETC of RD PDEs are detailed in Katz *et al.* (2021) and Lhachemi (2024). For parabolic PDEs with moving boundaries, static event-triggering using PDE backstepping is discussed in Rathnayake & Diagne (2022) and Koga *et al.* (2023), requiring full-state measurements. Dynamic event-triggering in the same context is presented in Demir *et al.* (2024) and Rathnayake & Diagne (2024a), with Demir *et al.* (2024) proposing a full-state feedback design and Rathnayake & Diagne (2024a) an observer-based design. In Kang *et al.* (2023), the authors introduce a full-state feedback dynamic ETC method for nonlinear RD PDEs. For RD PDEs with input delays full-state feedback ETC design is developed in Koudohode *et al.* (2024), whereas output feedback design can be found in Yuan *et al.* (2025). Globally exponentially stabilizing event-triggered gain-scheduling designs are developed in Karafyllis *et al.* (2021) for RD PDE with space- and time-dependent reactivity using PDE backstepping control.
- Research on ETC for hyperbolic PDEs spans a variety of interesting results. Static event-triggering mechanisms have been investigated in Espitia *et al.* (2016); Baudouin *et al.* (2019); Diagne & Karafyllis (2021); Koudohode *et al.* (2022) and Strecker *et al.* (2024), addressing linear hyperbolic

systems (Espitia *et al.*, 2016), damped wave equations (Baudouin *et al.*, 2019; Koudohode *et al.*, 2022), nonlinear hyperbolic PDEs in manufacturing (Diagne & Karafyllis, 2021) and 2×2 semilinear systems (Strecker *et al.*, 2024). These approaches generally rely on full-state feedback, except for Espitia *et al.* (2016), which uses output feedback. In contrast, dynamic event-triggering is the focus of Espitia *et al.* (2020); Espitia (2020); Wang & Krstic (2021); Espitia *et al.* (2022a); Wang & Krstic (2022a); Espitia *et al.* (2022b); Wang & Krstic (2022b, 2023b) and Zhang & Yu (2024), which study the case of 2×2 linear hyperbolic PDEs (Espitia, 2020; Wang & Krstic, 2021, 2022a,b, 2023b) and 4×4 systems (Espitia *et al.*, 2022a,b; Zhang & Yu, 2024). The extension of event-triggered gain scheduling to 2×2 hyperbolic PDEs with space- and time-dependent coefficient is established (Auriol & Espitia, 2024).

Periodic ETC and self-triggered control for finite-dimensional systems. Reducing the computational load of ETC, which requires continuous monitoring of the event-triggering mechanism for control updates, periodic event-triggered control (PETC) only checks for event-trigger conditions at fixed intervals–periodically–to decide whether an update of the control signal is necessary. This periodic checking reduces the need for constant state monitoring, improving efficiency while still addressing control needs when required (Heemels *et al.*, 2013). Numerous studies have explored the application of PETC to ODE systems in the recent years (Borgers *et al.*, 2018; Fu & Mazo, 2018; Linsenmayer *et al.*, 2019; Wang *et al.*, 2020; Seidel *et al.*, 2024) since the pioneering work of Heemels & Donkers (2013) and Heemels *et al.* (2013). Taking PETC a step further, self-triggered control (STC) eliminates the need for periodic updates by determining the next time an update is required, based on the system's state at current and/or past triggering instants. This approach not only enhances efficiency but also reduces communication and computational overhead. Notable results on STC for ODE systems include Mazo & Tabuada (2008); Mazo *et al.* (2009); Anta & Tabuada (2010); Yang *et al.* (2019); Yi *et al.* (2019); Wan *et al.* (2021) and Cao *et al.* (2023).

Periodic ETC and STC for infinite-dimensional systems. A few studies have explored STC and/or PETC strategies for infinite-dimensional systems, including Selivanov & Fridman (2016); Wakaiki & Sano (2020, 2022); Rathnayake & Diagne (2024b) and Rathnayake & Diagne (2023). In Selivanov & Fridman (2016), the authors propose a PETC approach for a network of semilinear diffusion PDEs with in-domain actuation and distributed or point measurements. Using semigroup theory, Wakaiki & Sano (2020) develops a full-state feedback PETC mechanism for infinite-dimensional systems with unbounded control operators, while Wakaiki & Sano (2022) introduces a full-state feedback STC approach for systems with bounded control operators. In Rathnayake & Diagne (2024b), the first PETC and STC PDE backstepping boundary control design is presented for a class of RD PDEs using boundary measurements. The observer-based PETC extension to diffusion PDEs with moving boundaries, including the one-phase Stefan problem, is discussed in Rathnayake & Diagne (2023). The extension of PDE backstepping PETC and STC to 2×2 hyperbolic PDEs with anti-collocated boundary actuation and sensing is achieved in Somathilake *et al.* (2025).

PETC and STC are well-suited for digital implementations, as the triggering conditions or determination of the next event time can be efficiently processed using standard time-sliced digital software.

Contributions. This work presents observer-based ETC, PETC and STC designs for a class of RD systems, ensuring global exponential stability (GES) through a dynamic triggering mechanism based on the backstepping method. Similar results have been proposed for 2×2 coupled hyperbolic PDEs in Rathnayake & Diagne (2026). Through time regularization, our approach allows for the explicit selection of a suitable MDT $\tau > 0$, inherently ensuring Zeno-free behaviour. The idea of applying time regularization to guarantee a MDT and incorporating a switching dynamic variable has been explored before in the context of PDE-based ETC. For instance, Katz *et al.* (2021) and Kang *et al.* (2023) present dynamic ETC schemes for parabolic PDEs, relying on a modal decomposition approach and linear matrix inequality techniques. Nevertheless, our approach draws its main inspiration from Dolk & Heemels (2017) and Dolk *et al.* (2017a,b), which develop dynamic ETC methods for NCS modelled by ODEs, making use of switching dynamic variables. We construct a switching dynamic variable that remains nondecreasing between the last event at time t_j and the MDT, after which it is permitted to decrease until the next event occurs at the zero-crossing of the dynamic variable. By carefully designing the switching dynamics, selecting appropriate initial conditions at each event and determining a suitable MDT $\tau > 0$ based on a state-independent dynamic reset variable, we ensure that the Lyapunov function remains dissipative along the closed-loop system dynamics, even with the event-triggered application of the control input using a zero-order hold. Previous studies using a similar approach—integrating dynamic triggering conditions with PDE backstepping—have achieved global exponential convergence rather than GES (Espitia *et al.*, 2020; Espitia, 2020; Wang & Krstic, 2021; Espitia *et al.*, 2022a; Rathnayake *et al.*, 2022a; Wang & Krstic, 2022a; Espitia *et al.*, 2022b; Rathnayake *et al.*, 2022b; Wang & Krstic, 2022b, 2023a,b; Demir *et al.*, 2024; Zhang & Yu, 2024; Rathnayake & Diagne, 2024a; Rathnayake *et al.*, 2025; Zhang *et al.*, 2025). In these works, the enforcement of an MDT to prevent Zeno behaviour depends on a nonzero initial condition for the dynamic event trigger, which inherently obstructs GES. Unlike existing static (Espitia *et al.*, 2021; Rathnayake & Diagne, 2022; Koga *et al.*, 2023; Koudohode *et al.*, 2024; Strecker *et al.*, 2024) and dynamic event-triggering (Espitia *et al.*, 2020; Espitia, 2020; Wang & Krstic, 2021; Espitia *et al.*, 2022a; Rathnayake *et al.*, 2022a; Wang & Krstic, 2022a; Espitia *et al.*, 2022b; Rathnayake *et al.*, 2022b; Wang & Krstic, 2022b, 2023a,b; Demir *et al.*, 2024,?; Zhang & Yu, 2024; Rathnayake & Diagne, 2024a,b; Rathnayake *et al.*, 2025; Somathilake *et al.*, 2025; Zhang *et al.*, 2025) strategies for PDE backstepping control, which require continuous measurements, our STC framework operates using only event-triggered measurements.

In summary, we propose three main contributions in this work. First, we develop a novel observer-based PDE backstepping continuous-time event-triggered control (CETC) for a class of RD PDEs, by constructing a novel dynamic event-trigger as well as a novel Lyapunov functional to ensure GES of the resulting closed-loop system. Second, we introduce a novel observer-based PETC for a class of RD PDEs, guaranteeing GES. Finally, we present a novel full-state feedback STC for a class of RD PDEs, which ensures GES and operates using only measurements at the triggering instants.

The paper is organized as follows. Section 2 states problem formulation and preliminaries. In Sections 3–5, we introduce the CETC, PETC and STC designs, respectively. Simulation results are presented in Section 6, and concluding remarks are provided in Section 7.

1.1. *Notations*

Let $\mathbb{R}$ be the set of real numbers, $\mathbb{R}_{>0}$ be the set of positive real numbers, and $\mathbb{R}_{\geq 0}$ be the set of non-negative real numbers including zero. Let $\mathbb{N}$ be the set of natural numbers including 0, and let $\mathbb{N}_{>0}$ be the set of natural numbers greater than 0. By $L^2(0,1)$, we denote the equivalence class of Lebesgue measurable functions $f : [0,1] \to \mathbb{R}$ such that $\|f\|_{L^2((0,1);\mathbb{R})} = (\int_0^1 |f(x)|^2 dx)^{1/2} < \infty$. Define $C^0(I; L^2((0,1);\mathbb{R}))$ as the space of continuous functions $u(\cdot,t)$ for an interval $I \subseteq \mathbb{R}_{>0}$ such that $I \ni t \to u(\cdot,t) \in L^2((0,1);\mathbb{R})$.

2. Preliminaries and problem formulation

Consider the following one-dimensional RD PDE with constant coefficients:

$$u_t(x,t) = \varepsilon u_{xx}(x,t) + \lambda u(x,t), \forall x \in (0,1), \tag{2.1}$$

$$u_x(0,t) = 0, \tag{2.2}$$

$$u_x(1,t) = -qu(1,t) + U_j, \tag{2.3}$$

for all $t \in (t_j, t_{j+1}), j \in \mathbb{N}$. The set $\{t_j\}_{j\in\mathbb{N}}$ is the set of control update times that will later be characterized under CETC, periodic event-triggered control (PETC) and STC strategies. The initial condition is such that $u[0] \in L^2(0,1)$, and the coefficients ε, λ, q are all positive. The input U_j is held constant for all $t \in [t_j, t_{j+1}), j \in \mathbb{N}$.

Assumption 1. The parameters $q, \lambda, \varepsilon > 0$ satisfy the following relation:

$$q > \frac{\lambda}{2\varepsilon} + \frac{1}{2}. \tag{2.4}$$

Assumption 1 plays a crucial role in guaranteeing the stability of the target system in PDE backstepping. This is because we deliberately exclude the use of the signal $u(1,t)$ from the nominal control law. Such exclusion is essential in the context of ETC due to the challenges in establishing a meaningful bound on the rate of change of $u(1,t)$. Additionally, it is important to highlight that an eigenfunction expansion of the solution to (2.1)–(2.3) under zero input ($U(t) = 0$) reveals that the system is unstable whenever $\lambda > \varepsilon\pi^2/4$, regardless of the value of $q > 0$.

In Rathnayake *et al.* (2022a), an observer is proposed for the system (2.1)–(2.3), utilizing $u(0,t)$ as the available boundary measurement. This leads to an anti-collocated configuration for sensing and actuation. The corresponding design is outlined below. The observer state $\hat{u}$ satisfies

$$\hat{u}_t(x,t) = \varepsilon\hat{u}_{xx}(x,t) + \lambda\hat{u}(x,t) + p_1(x)\tilde{u}(0,t), \text{ for all } x \in (0,1), \tag{2.5}$$

$$\hat{u}_x(0,t) = p_{10}\tilde{u}(0,t), \tag{2.6}$$

$$\hat{u}_x(1,t) = -q\hat{u}(1,t) + U_j, \tag{2.7}$$

for all $t \in (t_j, t_{j+1}), j \in \mathbb{N}$ with $\hat{u}[0] \in L^2(0,1)$, where

$$\tilde{u}(x,t) := u(x,t) - \hat{u}(x,t), \tag{2.8}$$

is the observer error satisfying

$$\tilde{u}_t(x,t) = \varepsilon\tilde{u}_{xx}(x,t) + \lambda\tilde{u}(x,t) - p_1(x)\tilde{u}(0,t), \text{ for all } x \in (0,1), \tag{2.9}$$

$$\tilde{u}_x(0,t) = -p_{10}\tilde{u}(0,t), \tag{2.10}$$

$$\tilde{u}_x(1,t) = -q\tilde{u}(1,t), \tag{2.11}$$

for all $t > 0$. The terms $p_1(x)$ and p_{10} are the observer gains satisfying

$$p_1(x) = \varepsilon P_y(x, 0), \forall x \in (0, 1), \tag{2.12}$$

$$p_{10} = -\frac{\lambda}{2\varepsilon}, \tag{2.13}$$

where

$$P(x, y) = \frac{q\frac{\lambda}{\varepsilon}}{\sqrt{\frac{\lambda}{\varepsilon} + q^2}} \int_0^{x-y} e^{\frac{-q\tau}{2}} I_0\Big(\sqrt{\frac{\lambda}{\varepsilon}(2 - x - y)(x - y - \tau)}\Big) \sinh\left(\frac{\sqrt{\frac{\lambda}{\varepsilon} + q^2}}{2}\tau\right) d\tau$$
$$- \frac{\lambda}{\varepsilon}(1 - y) \frac{I_1\Big(\sqrt{\frac{\lambda}{\varepsilon}\big((1 - y)^2 - (1 - x)^2\big)}\Big)}{\sqrt{\frac{\lambda}{\varepsilon}\big((1 - y)^2 - (1 - x)^2\big)}}, \tag{2.14}$$

for all $0 \leq y \leq x \leq 1$. The observer gains (2.12)–(2.14) are determined using the PDE backstepping technique equipped with the Volterra transformation

$$\tilde{u}(x, t) = \tilde{w}(x, t) - \int_0^x P(x, y)\tilde{w}(y, t)dy, \tag{2.15}$$

defined in the domain $0 \leq y \leq x \leq 1$. By applying the transformation (2.15), (2.14) along with the observer gains (2.12)–(2.14), the observer error system (2.9)–(2.11) is converted into the following observer error target system

$$\tilde{w}_t(x, t) = \varepsilon \tilde{w}_{xx}(x, t), \text{ for all } x \in (0, 1), \tag{2.16}$$

$$\tilde{w}_x(0, t) = 0, \tag{2.17}$$

$$\tilde{w}_x(1, t) = -q\tilde{w}(1, t), \tag{2.18}$$

valid for all $t > 0$. The inverse transformation of (2.15) is

$$\tilde{w}(x, t) = \tilde{u}(x, t) + \int_0^x Q(x, y)\tilde{w}(y, t)dy, \tag{2.19}$$

where

$$Q(x, y) = \frac{q\frac{\lambda}{\varepsilon}}{\sqrt{-\frac{\lambda}{\varepsilon} + q^2}} \int_0^{x-y} e^{\frac{-q\tau}{2}} J_0\Big(\sqrt{\frac{\lambda}{\varepsilon}(2 - x - y)(x - y - \tau)}\Big) \sinh\left(\frac{\sqrt{-\frac{\lambda}{\varepsilon} + q^2}}{2}\tau\right) d\tau$$
$$- \frac{\lambda}{\varepsilon}(1 - y) \frac{J_1\Big(\sqrt{\frac{\lambda}{\varepsilon}\big((1 - y)^2 - (1 - x)^2\big)}\Big)}{\sqrt{\frac{\lambda}{\varepsilon}\big((1 - y)^2 - (1 - x)^2\big)}}, \tag{2.20}$$

for all $0 \leq y \leq x \leq 1$.

In Rathnayake *et al.* (2022a), the following sampled-data boundary control law is derived

$$U_j = \int_0^1 k(y)\hat{u}(y, t_j)dy, \tag{2.21}$$

which can be implemented alongside a suitable event-triggering mechanism, for all $t \in [t_j, t_{j+1}), j \in \mathbb{N}$ where $k(y)$ is the control gain

$$k(y) := rK(1, y) + K_x(1, y), \tag{2.22}$$

with

$$r = q - \frac{\lambda}{2\varepsilon}, \tag{2.23}$$

and

$$K(x, y) = -\frac{\lambda}{\varepsilon} x \frac{I_1\left(\sqrt{\lambda(x^2 - y^2)/\varepsilon}\right)}{\sqrt{\lambda(x^2 - y^2)/\varepsilon}}, \tag{2.24}$$

for all $0 \leq y \leq x \leq 1$. The control gain $k(y)$ is determined using the PDE backstepping method, which incorporates the Volterra transformation given by

$$\hat{w}(x, t) = \hat{u}(x, t) - \int_0^x K(x, y)\hat{u}(y, t)\, dy, \tag{2.25}$$

for all $0 \leq y \leq x \leq 1$. Applying the transformation (2.25), (2.24), and considering the control input defined in (2.21)–(2.24), the observer is converted into the following observer target system

$$\hat{w}_t(x, t) = \varepsilon \hat{w}_{xx}(x, t) + g(x)\tilde{w}(0, t), \text{ for all } x \in (0, 1), \tag{2.26}$$

$$\hat{w}_x(0, t) = p_{10}\tilde{w}(0, t), \tag{2.27}$$

$$\hat{w}_x(1, t) = -r\hat{w}(1, t) + d(t), \tag{2.28}$$

for all $t \in (t_j, t_{j+1}), j \in \mathbb{N}$, where

$$g(x) = p_1(x) - \frac{\lambda}{2}K(x, 0) - \int_0^x K(x, y)p_1(y)dy, \tag{2.29}$$

for all $x \in (0, 1)$, and

$$d(t) = \int_0^1 k(y)\left(\hat{u}(y, t_j) - \hat{u}(y, t)\right)dy, \tag{2.30}$$

for all $t \in [t_j, t_{j+1}), j \in \mathbb{N}$. The term $d(t)$ represents the difference between the sampled-data control input and the corresponding continuous-time control input. The inverse transformation of (2.25) is

$$\hat{u}(x,t) = \hat{w}(x,t) + \int_0^x L(x,y)\hat{w}(y,t)dy, \tag{2.31}$$

where $L(x,y)$ satisfies

$$L(x,y) = -\frac{\lambda}{\varepsilon} x \frac{J_1\left(\sqrt{\lambda(x^2-y^2)/\varepsilon}\right)}{\sqrt{\lambda(x^2-y^2)/\varepsilon}}, \tag{2.32}$$

for all $0 \le y \le x \le 1$. In the following sections, we introduce different triggering mechanisms—continuous-time event-triggering, periodic event-triggering and self-triggering—to determine the control update instants $\{t_j\}_{j\in\mathbb{N}}$ that guarantee GES.

The following proposition establishes the well-posedness of the closed-loop system (2.1)–(2.8), (2.12)–(2.14), (2.21)–(2.24) considering piecewise constant inputs between successive sampling instants.

Proposition 1 (Rathnayake *et al.*, 2022a). For every $u[t_j], \hat{u}[t_j] \in L^2(0,1)$, there exist unique solutions $u, \hat{u} : [t_j, t_{j+1}] \times [0,1] \to \mathbb{R}$ between two time instants t_j and t_{j+1} such that $u, \hat{u} \in C^0([t_j, t_{j+1}]; L^2(0,1)) \cap C^1((t_j, t_{j+1}) \times [0,1])$ with $u[t], \hat{u}[t] \in C^2([0,1])$, which satisfy (2.2), (2.3), (2.6), (2.7), (2.13), (2.14), (2.21)–(2.24) for all $t \in (t_j, t_{j+1}]$ and (2.1), (2.5), (2.12), (2.14) for all $t \in (t_j, t_{j+1}]$, $x \in (0,1)$.

3. Continuous-time ETC

In this section, we introduce an event-triggering strategy that updates control actions only at specific instances while necessitating continuous monitoring of the triggering function to identify these events. This approach is referred to as *continuous-time event-triggering*. The proposed mechanism involves constructing a switching dynamic variable that absorbs the effects of control input sampling, ensuring the GES of the closed-loop system. The specifics of the design are outlined below.

Definition 1. Let $\eta > 0$ and $\beta_1, \beta_2, \beta_3 \ge 0$ satisfying $\beta_1 + \beta_2 + \beta_3 > 0$ be free event-trigger parameters, and let $\rho > 0$ be an event-trigger parameter to be determined later. The set of CETC event times $I = \{t_j\}_{j\in\mathbb{N}}$ is defined as an increasing sequence satisfying $\lim_{j\to\infty} t_j = +\infty$ according to the rule:

$$t_{j+1} = \inf\left\{t \ge t_j + \tau \mid m(t) < 0\right\}, \tag{3.1}$$

with $t_0 = 0$, where $\tau > 0$ represents the MDT to be selected, and the function $m(t)$ evolves as follows:

$$\dot{m}(t) = -\eta m(t) + \beta_1 \|\hat{u}[t]\|^2 + \beta_2 \hat{u}^2(1,t) + \beta_3 \tilde{u}^2(0,t), \tag{3.2}$$

for all $t \in (t_j, t_j + \tau), j \in \mathbb{N}$, and

$$\dot{m}(t) = -\eta m(t) - \rho d^2(t) + \beta_1 \|\hat{u}[t]\|^2 + \beta_2 \hat{u}^2(1,t) + \beta_3 \tilde{u}^2(0,t), \tag{3.3}$$

for all $t \in (t_j + \tau, t_{j+1}), j \in \mathbb{N}$, where the initial condition is $m(t_0) = m(0) = 0$. Moreover, at each event time, $m(t_j) \geq 0$ for $j \in \mathbb{N}_{>0}$ is to be appropriately selected. At $t = t_j + \tau$, it is chosen that $m\big((t_j + \tau)^-\big) = m(t_j + \tau), j \in \mathbb{N}$ ensuring continuity. In (3.3), the term $d(t)$ is given by (2.30).

Prior to presenting the main results under CETC, we first establish several auxiliary results stated in Lemmas 1-3. These results play a crucial role in proving the main findings outlined in Theorem 1.

Lemma 1. Consider a CETC approach of the form (2.21)–(2.24), (3.1)–(3.3), which generates a set of increasing event times $I = \{t_j\}_{j\in\mathbb{N}}$ with $t_0 = 0$, satisfying $\lim_{j\to\infty} t_j = +\infty$. Then, for the dynamic variable $m(t)$ governed by (3.2), (3.3), initialized at $m(0) = 0$, with $m(t_j) \geq 0$ for $j \in \mathbb{N}_{>0}$ to be specified, and satisfying $m\big((t_j + \tau)^-\big) = m(t_j + \tau)$ for $j \in \mathbb{N}$, where $\tau > 0$ is the MDT to be selected, it follows that $m(t) \geq 0$ for all $t > 0$.

Proof. Let $m(t_j)$ for $j \in \mathbb{N}_{>0}$ be such that $m(t_j) \geq 0$. Then, from (3.2), it follows that

$$m(t) = e^{-\eta(t-t_j)}m(t_j) + \int_{t_j}^{t} e^{-\eta(t-\xi)}\Big(\beta_1\|\hat{u}[\xi]\|^2 + \beta_2\hat{u}^2(1,\xi) + \beta_3\tilde{u}^2(0,\xi)\Big)d\xi \geq 0, \tag{3.4}$$

for all $t \in (t_j, t_j + \tau]$ and $j \in \mathbb{N}$. Once t reaches $t = t_j + \tau$, the evolution of $m(t)$ follows (3.3), and an event is triggered at $t = t_{j+1}$ when $m(t_{j+1}) = 0$ based on (3.1), ensuring that $m(t) \geq 0$ for all $t \in (t_j + \tau, t_{j+1})$, where $j \in \mathbb{N}$. By following this reasoning iteratively, starting with $m(t_0) = 0$ and letting $m(t_j) \geq 0, j \in \mathbb{N}_{>0}$, it can be concluded that $m(t) \geq 0$ for all $t > 0$. □

Lemma 2. Consider a set of increasing event times $I = \{t_j\}_{j\in\mathbb{N}}$ with $t_0 = 0$, satisfying $\lim_{j\to\infty} t_j = +\infty$. Then, for $d(t)$ given by (2.30), it holds that

$$\dot{d}^2(t) \leq \rho_1 d^2(t) + \alpha_1\|\hat{u}[t]\|^2 + \alpha_2\hat{u}^2(1,t) + \alpha_3\tilde{u}^2(0,t) \tag{3.5}$$

for all $t \in (t_j, t_{j+1}), j \in \mathbb{N}$, where

$$\rho_1 = 4\varepsilon^2 k^2(1), \tag{3.6}$$

$$\alpha_1 = 4\int_0^1 \big(\varepsilon k''(y) + \varepsilon k(1)k(y) + \lambda k(y)\big)^2 dy, \tag{3.7}$$

$$\alpha_2 = 4\big(\varepsilon q k(1) + \varepsilon k'(1)\big)^2, \tag{3.8}$$

$$\alpha_3 = 4\Big(\frac{\lambda k(0)}{2} + \int_0^1 k(y)p_1(y)dy\Big)^2, \tag{3.9}$$

with $p_1(y)$ and $k(y)$ given by (2.12) and (2.22), respectively.

The proof is similar to that of Rathnayake *et al.* (2022a), and hence omitted.

LEMMA 3. Consider a set of increasing event times $I = \{t_j\}_{j\in\mathbb{N}}$ with $t_0 = 0$, satisfying $\lim_{j\to\infty} t_j = +\infty$. Further, consider a piecewise right continuous function $f(t) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ that satisfies

$$\dot{f}(t) = \begin{cases} -a_2 f^2(t) - a_1 f(t) - a_0, & \forall t \in (t_j, t_j + \tau), \\ 0, & \forall t \in (t_j + \tau, t_{j+1}), \end{cases} \tag{3.10}$$

for $j \in \mathbb{N}$, where $\tau > 0$ is the MDT, and $a_0, a_1, a_2 > 0$, with $f(t_j) = \omega_1 > 0$ and $f\big((t_j + \tau)^-\big) = f(t_j + \tau) = \omega_0 > 0$ and $\omega_0 < \omega_1$. Then, the following results hold:

R1: The MDT $\tau > 0$ is

$$\tau = \int_{\omega_0}^{\omega_1} \frac{1}{a_2 s^2 + a_1 s + a_0} ds. \tag{3.11}$$

R2: For $j \in \mathbb{N}$, the function $f(t)$ strictly decreases when $t \in [t_j, t_j + \tau)$ and remains constant when $t \in [t_j + \tau, t_{j+1})$. Specifically, the function $f(t)$ reduces from $f(t_j) = \omega_1$ to $f(t_j + \tau) = \omega_0$ for all $t \in [t_j, t_j + \tau)$, and remains at $f(t) = \omega_0$ for all $t \in [t_j + \tau, t_{j+1})$.

Proof. By considering (3.10), we can readily establish that the duration required for $f(t)$ to transition from $\omega_1 > 0$ to $\omega_0 \geq 0$, denoted as τ, is given by (3.11). Additionally, from (3.10), it follows that for all $t \in (t_j, t_j + \tau)$, the derivative $\dot{f}(t)$ remains negative whenever $f(t) \geq 0$, ensuring a strictly decreasing behaviour of $f(t)$. Consequently, $f(t)$ monotonically decreases over the interval $t \in [t_j, t_j + \tau)$, moving from $f(t_j) = \omega_1$ to $f(t_j + \tau) = \omega_0$. However, since $\dot{f}(t) = 0$ for all $t \in (t_j + \tau, t_{j+1})$, the function $f(t)$ remains constant at $f(t) = \omega_0$ throughout the interval $t \in [t_j + \tau, t_{j+1})$. □

REMARK 1. By applying partial fraction expansion, the MDT given in (3.11) can be expressed as

$$\tau = \begin{cases} \dfrac{\omega_1 - \omega_0}{a_2\left(\omega_1 + \frac{a_1}{2a_2}\right)\left(\omega_0 + \frac{a_1}{2a_2}\right)}, & \text{if } \Delta = 0, \\ \dfrac{1}{\sqrt{\Delta}} \ln\left(1 + \dfrac{\frac{\sqrt{\Delta}}{a_2}(\omega_1 - \omega_0)}{\left(\omega_1 + \frac{a_1 + \sqrt{\Delta}}{2a_2}\right)\left(\omega_0 + \frac{a_1 - \sqrt{\Delta}}{2a_2}\right)}\right), & \text{if } \Delta > 0, \\ \dfrac{2}{\sqrt{-\Delta}} \tan^{-1}\left(\dfrac{\frac{2a_2}{\sqrt{-\Delta}}(\omega_1 - \omega_0)}{1 - \frac{4a_2^2}{\Delta}\left(\omega_1 + \frac{a_1}{2a_2}\right)\left(\omega_0 + \frac{a_1}{2a_2}\right)}\right), & \text{if } \Delta < 0, \end{cases} \tag{3.12}$$

where

$$\Delta := a_1^2 - 4a_0 a_2. \tag{3.13}$$

As will be discussed later, the parameters $a_1, a_2 > 0$ and $\omega_1 > \omega_0 > 0$ are freely selectable, while $a_0 > 0$ must be chosen appropriately.

The following theorem establishes the GES of the closed-loop system (2.1)–(2.3), (2.5)–(2.14) under the CETC framework defined by (2.21)–(2.24), (3.1)–(3.3).

THEOREM 1. Consider the CETC approach (2.21)–(2.24), (3.1)–(3.3), which generates a set of increasing event times $I = \{t_j\}_{j\in\mathbb{N}}$ with $t_0 = 0$, satisfying $\lim_{j\to\infty} t_j = +\infty$. For every $u[0], \hat{u}[0] \in L^2(0,1)$, there exist unique solutions $u, \hat{u} : \mathbb{R}_+ \times [0,1] \to \mathbb{R}$ such that $u, \hat{u} \in C^0(\mathbb{R}_+; L^2(0,1) \cap C^1(J \times [0,1])$ with $u[t] \in C^2([0,1])$ which satisfy (2.2), (2.3), (2.6), (2.7), (2.13), (2.14), (2.21)–(2.24) for all $t > 0$ and (2.1), (2.5), (2.12), (2.14) for all $t > 0, x \in (0,1)$, where $J = \mathbb{R}_+ \backslash I$. Let the event-trigger parameters $\beta_1, \beta_2, \beta_3 \geq 0$ be chosen such that $\beta_1 + \beta_2 + \beta_3 > 0$, and let $\eta > 0$. Further, let $a_1, a_2 > 0, \omega_1 > \omega_0 > 0$, and let $a_0 > 0$ be chosen such that

$$a_0 = \frac{\varepsilon \kappa_1 B}{2} + \frac{\rho_1}{a_2}, \tag{3.14}$$

where $\rho_1 > 0$ is given by (3.6), and subject to Assumption 1, the parameters $B, \kappa_1 > 0$ are chosen such that

$$B\left(\varepsilon \min\left\{r - \frac{1}{2}, \frac{1}{2}\right\} - \frac{\varepsilon}{2\kappa_1} - \frac{5\lambda}{8\kappa_2} - \frac{\|g\|^2}{\kappa_3}\right) - 2\beta_1 \tilde{L}^2 - 4\beta_2 \check{L}^2 - 2\beta_2 - \frac{(2\alpha_1 \tilde{L}^2 + 4\alpha_2 \check{L}^2 + 2\alpha_2)}{a_2} > 0, \tag{3.15}$$

for some $\kappa_2, \kappa_3 > 0$, where $\alpha_1, \alpha_2 > 0$ are given by (3.7), (3.8), and

$$\tilde{L} = 1 + \left(\int_0^1 \int_0^x L^2(x,y)dydx\right)^{1/2}, \tag{3.16}$$

and

$$\check{L} = \left(\int_0^1 L^2(1,y)dy\right)^{1/2}, \tag{3.17}$$

with $L(x,y)$ given by (2.32). Let the event-trigger parameter $\rho > 0$ be chosen as

$$\rho = a_2\omega_0^2 + a_1\omega_0 + a_0. \tag{3.18}$$

Further, let the MDT $\tau > 0$ be chosen as in (3.11), and initial conditions of $m(t)$ satisfying (3.2), (3.3) be chosen as

$$m(t_0) = m(0) = 0, \tag{3.19}$$

$$m\big((t_j + \tau)^-\big) = m(t_j + \tau), \text{ for } j \in \mathbb{N}, \tag{3.20}$$

$$m(t_j) = \omega_0 d^2(t_j^-), \text{ for } j \in \mathbb{N}_{>0}. \tag{3.21}$$

Then, under the CETC approach (2.21)–(2.24), (3.1)–(3.3), the closed-loop system consisting of the plant (2.1)–(2.3) and the observer (2.5)–(2.14) is globally exponentially stable. More specifically, the following estimate holds:

$$\|u[t]\| + \|\hat{u}[t]\| \leq M_{etc} e^{-\upsilon^{\star}_{etc} t} \big(\|u[0]\| + \|\hat{u}[0]\| \big), \tag{3.22}$$

for all $t > 0$, where $M_{etc} > 0$ and $\upsilon^{\star}_{etc} > 0$ are some positive constants.

Proof. The well-posedness in the sense of Theorem 1 is obtained by applying Proposition 1 iteratively in between two events.

Let us consider the following Lyapunov candidate

$$V_2(t) := V_1(t) + f(t) d^2(t) + m(t), \tag{3.23}$$

with

$$V_1(t) := \frac{A}{2} \|\tilde{w}[t]\|^2 + \frac{B}{2} \|\hat{w}[t]\|^2, \tag{3.24}$$

where $\tilde{w}$ satisfies (2.16)–(2.18), $\hat{w}$ satisfies (2.26)–(2.30), $m(t)$ satisfies (3.2), (3.3), and $f(t)$ satisfies (3.10). Recall from Lemma 1 that $m(t) \geq 0$ for all $t > 0$, and note from Lemma 3 that $f(t) > 0$ for all $t \geq 0$. Let $B > 0$ be selected as in (3.15), and let $A > 0$ be selected such that

$$A\varepsilon \min \left\{ q - \frac{1}{2}, \frac{1}{2} \right\} - 5\Big(\frac{\lambda \kappa_2 B}{8} + \frac{\kappa_3 B}{4} + \frac{\beta_3}{2} + \frac{\alpha_3}{2a_2} \Big) > 0, \tag{3.25}$$

where α_3 is given by (3.9), $\kappa_2, \kappa_3 > 0$ are selected such that (3.15) holds, and $a_2 > 0$ is freely selected. Taking the time derivative of $V_1(t)$ for all $t \in (t_j, t_{j+1}), j \in \mathbb{N}$ and integrating by parts, we obtain that

$$\begin{aligned} \dot{V}_1(t) =& -A\varepsilon q \tilde{w}^2(1,t) - A\varepsilon \|\tilde{w}_x[t]\|^2 - r\varepsilon B \hat{w}^2(1,t) + \varepsilon B d(t) \hat{w}(1,t) \\ &+ \frac{\lambda B}{2} \hat{w}(0,t) \tilde{w}(0,t) - \varepsilon B \|\hat{w}_x[t]\|^2 + B \int_0^1 g(x) \hat{w}(x,t) dx \tilde{w}(0,t). \end{aligned} \tag{3.26}$$

From Young's and Cauchy-Schwarz inequalities, we can obtain that

$$\varepsilon B d(t) \hat{w}(1,t) \leq \frac{\varepsilon B}{2\kappa_1} \hat{w}^2(1,t) + \frac{\varepsilon \kappa_1 B}{2} d^2(t), \tag{3.27}$$

$$\frac{\lambda B}{2} \hat{w}(0,t) \tilde{w}(0,t) \leq \frac{\lambda B}{4\kappa_2} \hat{w}^2(0,t) + \frac{\lambda \kappa_2 B}{4} \tilde{w}^2(0,t), \tag{3.28}$$

$$B \int_0^1 g(x) \hat{w}(x,t) dx \tilde{w}(0,t) \leq \frac{\|g\|^2 B}{2\kappa_3} \|\hat{w}[t]\|^2 + \frac{\kappa_3 B}{2} \tilde{w}^2(0,t), \tag{3.29}$$

for all $t \geq 0$, for $\kappa_1, \kappa_2, \kappa_3 > 0$. From Agmon's and Young's inequalities, we have that

$$\tilde{w}^2(0,t) \leq \tilde{w}^2(1,t) + \|\tilde{w}[t]\|^2 + \|\tilde{w}_x[t]\|^2, \tag{3.30}$$

$$\hat{w}^2(0,t) \leq \hat{w}^2(1,t) + \|\hat{w}[t]\|^2 + \|\hat{w}_x[t]\|^2, \tag{3.31}$$

for all $t \geq 0$. Therefore, we can show from (3.26) using (3.27)–(3.31) that

$$\dot{V}_1(t) \leq -\Big(A\varepsilon q - \frac{\lambda\kappa_2 B}{4} - \frac{\kappa_3 B}{2}\Big)\tilde{w}^2(1,t) - \Big(A\varepsilon - \frac{\lambda\kappa_2 B}{4} - \frac{\kappa_3 B}{2}\Big)\|\tilde{w}_x[t]\|^2 + \frac{\varepsilon\kappa_1 B}{2}d^2(t) + \Big(\frac{\lambda\kappa_2 B}{4} + \frac{\kappa_3 B}{2}\Big)\|\tilde{w}[t]\|^2 - \Big(r\varepsilon B - \frac{\varepsilon B}{2\kappa_1} - \frac{\lambda B}{4\kappa_2}\Big)\hat{w}^2(1,t) - \Big(\varepsilon B - \frac{\lambda B}{4\kappa_2}\Big)\|\hat{w}_x[t]\|^2 + \Big(\frac{\lambda B}{4\kappa_2} + \frac{\|g\|^2 B}{2\kappa_3}\Big)\|\hat{w}[t]\|^2, \tag{3.32}$$

for all $t \in (t_j, t_{j+1}), j \in \mathbb{N}$. Now let us consider $V_2(t)$ given by (3.23) for all $t \in (t_j, t_j + \tau), j \in \mathbb{N}$.

For all $t \in (t_j, t_j + \tau), j \in \mathbb{N}$:

Taking the time derivative of (3.23) for all $t \in (t_j, t_j + \tau), j \in \mathbb{N}$, using Young's inequality and considering (3.2), (3.10), we can write that

$$\begin{aligned}\dot{V}_2(t) =& \dot{V}_1(t) + \dot{f}(t)d^2(t) + 2f(t)d(t)\dot{d}(t) + \dot{m}(t)\\ \leq& \dot{V}_1(t) + \dot{f}(t)d^2(t) + a_2 f^2(t)d^2(t) + \frac{1}{a_2}\dot{d}^2(t) + \dot{m}(t)\\ \leq& -\Big(A\varepsilon q - \frac{\lambda\kappa_2 B}{4} - \frac{\kappa_3 B}{2}\Big)\tilde{w}^2(1,t) - \Big(A\varepsilon - \frac{\lambda\kappa_2 B}{4} - \frac{\kappa_3 B}{2}\Big)\|\tilde{w}_x[t]\|^2 + \frac{\varepsilon\kappa_1 B}{2}d^2(t)\\ &+ \Big(\frac{\lambda\kappa_2 B}{4} + \frac{\kappa_3 B}{2}\Big)\|\tilde{w}[t]\|^2 - \Big(r\varepsilon B - \frac{\varepsilon B}{2\kappa_1} - \frac{\lambda B}{4\kappa_2}\Big)\hat{w}^2(1,t) - \Big(\varepsilon B - \frac{\lambda B}{4\kappa_2}\Big)\|\hat{w}_x[t]\|^2\\ &+ \Big(\frac{\lambda B}{4\kappa_2} + \frac{\|g\|^2 B}{2\kappa_3}\Big)\|\hat{w}[t]\|^2 - \big(a_2 f^2(t) + a_1 f(t) + a_0\big)d^2(t) + a_2 f^2(t)d^2(t) + \frac{\rho_1}{a_2}d^2(t)\\ &+ \frac{\alpha_1}{a_2}\|\hat{u}[t]\|^2 + \frac{\alpha_2}{a_2}\hat{u}^2(1,t) + \frac{\alpha_3}{a_2}\tilde{u}^2(0,t) - \eta m(t) + \beta_1\|\hat{u}[t]\|^2 + \beta_2\hat{u}^2(1,t) + \beta_3\tilde{u}^2(0,t).\end{aligned} \tag{3.33}$$

Using Young's and Cauchy-Schwarz inequalities, we can obtain from (2.31) that

$$\|\hat{u}[t]\|^2 \leq \tilde{L}^2\|\hat{w}[t]\|^2, \tag{3.34}$$

$$\hat{u}^2(1,t) \leq 2\hat{w}^2(1,t) + 2\check{L}^2\|\hat{w}[t]\|^2, \tag{3.35}$$

for all $t \geq 0$, where $\tilde{L}$ and $\check{L}$ are given by (3.16) and (3.17), respectively. Thus, considering (3.34), (3.35), noting from (2.15) that $\tilde{u}(0,t) = \tilde{w}(0,t)$ and using (3.30), we can rewrite (3.33) rearranging terms as

$$\dot{V}_2(t) \leq -\Big(A\varepsilon q - \frac{\lambda\kappa_2 B}{4} - \frac{\kappa_3 B}{2} - \beta_3 - \frac{\alpha_3}{a_2}\Big)\tilde{w}^2(1,t) - \Big(A\varepsilon - \frac{\lambda\kappa_2 B}{4} - \frac{\kappa_3 B}{2} - \beta_3 - \frac{\alpha_3}{a_2}\Big)\|\tilde{w}_x[t]\|^2 + \Big(\frac{\lambda\kappa_2 B}{4} + \frac{\kappa_3 B}{2} + \beta_3 + \frac{\alpha_3}{a_2}\Big)\|\tilde{w}[t]\|^2 - \Big(\varepsilon B - \frac{\lambda B}{4\kappa_2}\Big)\|\hat{w}_x[t]\|^2$$

$$
\begin{aligned}
&-\Big(r\varepsilon B-\frac{\varepsilon B}{2\kappa_1}-\frac{\lambda B}{4\kappa_2}-2\beta_2-\frac{2\alpha_2}{a_2}\Big)\hat{w}^2(1,t)\\
&+\Big(\frac{\lambda B}{4\kappa_2}+\frac{\|g\|^2B}{2\kappa_3}+\beta_1\tilde{L}^2+2\beta_2\check{L}^2+\frac{\alpha_1\tilde{L}^2+2\alpha_2\check{L}^2}{a_2}\Big)\|\hat{w}[t]\|^2\\
&-\Big(a_0-\frac{\varepsilon\kappa_1 B}{2}-\frac{\rho_1}{a_2}\Big)d^2(t)-a_1f(t)d^2(t)-\eta m(t),
\end{aligned}\tag{3.36}
$$

for all $t\in(t_j,t_j+\tau),j\in\mathbb{N}$. Recalling that a_0 is selected as in (3.14), we can simplify (3.36) to obtain

$$
\begin{aligned}
\dot{V}_2(t)\leq&-\Big(A\varepsilon q-\frac{\lambda\kappa_2B}{4}-\frac{\kappa_3B}{2}-\beta_3-\frac{\alpha_3}{a_2}\Big)\tilde{w}^2(1,t)-\Big(A\varepsilon-\frac{\lambda\kappa_2B}{4}-\frac{\kappa_3B}{2}-\beta_3-\frac{\alpha_3}{a_2}\Big)\|\tilde{w}_x[t]\|^2\\
&+\Big(\frac{\lambda\kappa_2B}{4}+\frac{\kappa_3B}{2}+\beta_3+\frac{\alpha_3}{a_2}\Big)\|\tilde{w}[t]\|^2-\Big(\varepsilon B-\frac{\lambda B}{4\kappa_2}\Big)\|\hat{w}_x[t]\|^2\\
&-\Big(r\varepsilon B-\frac{\varepsilon B}{2\kappa_1}-\frac{\lambda B}{4\kappa_2}-2\beta_2-\frac{2\alpha_2}{a_2}\Big)\hat{w}^2(1,t)\\
&+\Big(\frac{\lambda B}{4\kappa_2}+\frac{\|g\|^2B}{2\kappa_3}+\beta_1\tilde{L}^2+2\beta_2\check{L}^2+\frac{\alpha_1\tilde{L}^2+2\alpha_2\check{L}^2}{a_2}\Big)\|\hat{w}[t]\|^2-a_1f(t)d^2(t)-\eta m(t),
\end{aligned}\tag{3.37}
$$

for all $t\in(t_j,t_j+\tau),j\in\mathbb{N}$. From Poincaré Inequality, we have that

$$
-\|\tilde{w}_x[t]\|^2\leq\frac{1}{2}\tilde{w}^2(1,t)-\frac{1}{4}\|\tilde{w}[t]\|^2,\tag{3.38}
$$

$$
-\|\hat{w}_x[t]\|^2\leq\frac{1}{2}\hat{w}^2(1,t)-\frac{1}{4}\|\hat{w}[t]\|^2,\tag{3.39}
$$

for all $t\geq0$. Noting that $\big(A\varepsilon-\frac{\lambda\kappa_2B}{4}-\frac{\kappa_3B}{2}-\beta_3-\frac{\alpha_3}{a_2}\big)>0$ and $\big(\varepsilon B-\frac{\lambda B}{4\kappa_2}\big)>0$ from (3.25) and (3.15), respectively, and using (3.38) and (3.39) on (3.37), we can obtain that

$$
\begin{aligned}
\dot{V}_2\leq&-\Big(A\varepsilon(q-\frac{1}{2})-\frac{\lambda\kappa_2B}{8}-\frac{\kappa_3B}{4}-\frac{\beta_3}{2}-\frac{\alpha_3}{2a_2}\Big)\tilde{w}^2(1,t)\\
&-\Big(\frac{A\varepsilon}{4}-\frac{5\lambda\kappa_2B}{16}-\frac{5\kappa_3B}{8}-\frac{5\beta_3}{4}-\frac{5\alpha_3}{4a_2}\Big)\|\tilde{w}[t]\|^2\\
&-\Big(\varepsilon B(r-\frac{1}{2})-\frac{\varepsilon B}{2\kappa_1}-\frac{\lambda B}{8\kappa_2}-2\beta_2-\frac{2\alpha_2}{a_2}\Big)\hat{w}^2(1,t)\\
&-\Big(\frac{\varepsilon B}{4}-\frac{5\lambda B}{16\kappa_2}-\frac{\|g\|^2B}{2\kappa_3}-\beta_1\tilde{L}^2-2\beta_2\check{L}^2-\frac{(\alpha_1\tilde{L}^2+2\alpha_2\check{L}^2)}{a_2}\Big)\|\hat{w}[t]\|^2\\
&-a_1f(t)d^2(t)-\eta m(t),
\end{aligned}\tag{3.40}
$$

for all $t \in (t_j, t_j+\tau), j \in \mathbb{N}$. Note again from (3.25) and (3.15) that $A\varepsilon(q-\frac{1}{2}) - \frac{\lambda\kappa_2 B}{8} - \frac{\kappa_3 B}{4} - \frac{\beta_3}{2} - \frac{\alpha_3}{2a_2} > 0$ and $\varepsilon B(r-\frac{1}{2}) - \frac{\varepsilon B}{2\kappa_1} - \frac{\lambda B}{8\kappa_2} - 2\beta_2 - \frac{2\alpha_2}{a_2} > 0$. Further, from (3.25) and (3.15), we have that

$$\upsilon_{1,etc} := \frac{A\varepsilon}{4} - \frac{5\lambda\kappa_2 B}{16} - \frac{5\kappa_3 B}{8} - \frac{5\beta_3}{4} - \frac{5\alpha_3}{4a_2} > 0, \tag{3.41}$$

and

$$\upsilon_{2,etc} := \frac{\varepsilon B}{4} - \frac{5\lambda B}{16\kappa_2} - \frac{\|g\|^2 B}{2\kappa_3} - \beta_1\tilde{L}^2 - 2\beta_2\check{L}^2 - \frac{(\alpha_1\tilde{L}^2 + 2\alpha_2\check{L}^2)}{a_2} > 0. \tag{3.42}$$

Then, we can show for all $t \in (t_j, t_j+\tau), j \in \mathbb{N}$ that

$$\dot{V}_2(t) \le -2\upsilon^{\star}_{etc} V_2(t), \tag{3.43}$$

with

$$\upsilon^{\star}_{etc} = \min\left\{\frac{\upsilon_{1,etc}}{A}, \frac{\upsilon_{2,etc}}{B}, \frac{a_1}{2}, \frac{\eta}{2}\right\}. \tag{3.44}$$

For all $t \in (t_j+\tau, t_{j+1}), j \in \mathbb{N}$:

For all $t \in (t_j+\tau, t_{j+1}), j \in \mathbb{N}$, we have that $f(t) = \omega_0 > 0$ (see Lemma 3). Thus, taking the time derivative of (3.23) for all $t \in (t_j+\tau, t_{j+1}), j \in \mathbb{N}$, using Young's inequality, considering (3.3), (3.32), (3.34), (3.35), and noting from (2.15) that $\tilde{u}(0,t) = \tilde{w}(0,t)$, we can write that

$$\begin{aligned}
\dot{V}_2(t) =& \dot{V}_1(t) + 2\omega_0 d(t)\dot{d}(t) + \dot{m}(t) \\
\le& \dot{V}_1(t) + a_2\omega_0^2 d^2(t) + \frac{\dot{d}^2(t)}{a_2} + \dot{m}(t) \\
\le& -\left(A\varepsilon q - \frac{\lambda\kappa_2 B}{4} - \frac{\kappa_3 B}{2}\right)\tilde{w}^2(1,t) \\
&- \left(A\varepsilon - \frac{\lambda\kappa_2 B}{4} - \frac{\kappa_3 B}{2}\right)\|\tilde{w}_x[t]\|^2 + \frac{\varepsilon\kappa_1 B}{2} d^2(t) + \left(\frac{\lambda\kappa_2 B}{4} + \frac{\kappa_3 B}{2}\right)\|\tilde{w}[t]\|^2 \\
&- \left(r\varepsilon B - \frac{\varepsilon B}{2\kappa_1} - \frac{\lambda B}{4\kappa_2}\right)\hat{w}^2(1,t) - \left(\varepsilon B - \frac{\lambda B}{4\kappa_2}\right)\|\hat{w}_x[t]\|^2 + \left(\frac{\lambda B}{4\kappa_2} + \frac{\|g\|^2 B}{2\kappa_3}\right)\|\hat{w}[t]\|^2 \\
&+ a_2\omega_0^2 d^2(t) + \frac{\rho_1}{a_2} d^2(t) + \frac{(\alpha_1\tilde{L}^2 + 2\alpha_2\check{L}^2)}{a_2}\|\hat{w}[t]\|^2 + \frac{2\alpha_2}{a_2}\hat{w}^2(1,t) + \frac{\alpha_3}{a_2}\tilde{w}^2(0,t) - \eta m(t) \\
&- \rho d^2(t) + (\beta_1\tilde{L}^2 + 2\beta_2\check{L}^2)\|\hat{w}[t]\|^2 + 2\beta_2\hat{w}^2(1,t) + \beta_3\tilde{w}^2(0,t).
\end{aligned} \tag{3.45}$$

Using (3.30) and rearranging the terms of (3.45), we can obtain

$$\begin{aligned}\dot{V}_2(t) \leq &-\Big(A\varepsilon q - \frac{\lambda\kappa_2 B}{4} - \frac{\kappa_3 B}{2} - \beta_3 - \frac{\alpha_3}{a_2}\Big)\tilde{w}^2(1,t) - \Big(A\varepsilon - \frac{\lambda\kappa_2 B}{4} - \frac{\kappa_3 B}{2} - \beta_3 - \frac{\alpha_3}{a_2}\Big)\|\tilde{w}_x[t]\|^2 \\ &+ \Big(\frac{\lambda\kappa_2 B}{4} + \frac{\kappa_3 B}{2} + \beta_3 + \frac{\alpha_3}{a_2}\Big)\|\tilde{w}[t]\|^2 - \Big(\varepsilon B - \frac{\lambda B}{4\kappa_2}\Big)\|\hat{w}_x[t]\|^2 \\ &- \Big(r\varepsilon B - \frac{\varepsilon B}{2\kappa_1} - \frac{\lambda B}{4\kappa_2} - 2\beta_2 - \frac{2\alpha_2}{a_2}\Big)\hat{w}^2(1,t) \\ &+ \Big(\frac{\lambda B}{4\kappa_2} + \frac{\|g\|^2 B}{2\kappa_3} + \beta_1\tilde{L}^2 + 2\beta_1\check{L}^2 + \frac{\alpha_1\tilde{L}^2 + 2\alpha_2\check{L}^2}{a_2}\Big)\|\hat{w}[t]\|^2 \\ &- \Big(\rho - \frac{\varepsilon\kappa_1 B}{2} - \frac{\rho_1}{a_2} - a_2\omega_0^2\Big)d^2(t) - \eta m(t), \end{aligned} \tag{3.46}$$

for all $t \in (t_j + \tau, t_{j+1}), j \in \mathbb{N}$. Recalling that $\rho > 0$ is selected as in (3.18) with a_0 given by (3.14), we can simplify (3.46) to obtain

$$\begin{aligned}\dot{V}_2(t) \leq &-\Big(A\varepsilon q - \frac{\lambda\kappa_2 B}{4} - \frac{\kappa_3 B}{2} - \beta_3 - \frac{\alpha_3}{a_2}\Big)\tilde{w}^2(1,t) - \Big(A\varepsilon - \frac{\lambda\kappa_2 B}{4} - \frac{\kappa_3 B}{2} - \beta_3 - \frac{\alpha_3}{a_2}\Big)\|\tilde{w}_x[t]\|^2 \\ &+ \Big(\frac{\lambda\kappa_2 B}{4} + \frac{\kappa_3 B}{2} + \beta_3 + \frac{\alpha_3}{a_2}\Big)\|\tilde{w}[t]\|^2 - \Big(\varepsilon B - \frac{\lambda B}{4\kappa_2}\Big)\|\hat{w}_x[t]\|^2 \\ &- \Big(r\varepsilon B - \frac{\varepsilon B}{2\kappa_1} - \frac{\lambda B}{4\kappa_2} - 2\beta_2 - \frac{2\alpha_2}{a_2}\Big)\hat{w}^2(1,t) \\ &+ \Big(\frac{\lambda B}{4\kappa_2} + \frac{\|g\|^2 B}{2\kappa_3} + \beta_1\tilde{L}^2 + 2\beta_1\check{L}^2 + \frac{\alpha_1\tilde{L}^2 + 2\alpha_2\check{L}^2}{a_2}\Big)\|\hat{w}[t]\|^2 \\ &- a_1\omega_0 d^2(t) - \eta m(t), \end{aligned} \tag{3.47}$$

for all $t \in (t_j + \tau, t_{j+1}), j \in \mathbb{N}$. Then, following steps similar to (3.38)–(3.42), we can show for all $t \in (t_j + \tau, t_{j+1}), j \in \mathbb{N}$ that

$$\dot{V}_2(t) \leq -2\upsilon_{etc}^{\star} V_2(t), \tag{3.48}$$

where $\upsilon_{etc}^{\star} > 0$ is given by (3.44). Recall that $m(t_j + \tau)$ is chosen such that $m\big((t_j+\tau)^-\big) = m(t_j+\tau)$, and from Lemma 3 that $f\big((t_j+\tau)^-\big) = f(t_j+\tau) = \omega_0$. Further, note that $d\big((t_j+\tau)^-\big) = d(t_j+\tau)$ and that $\|\hat{w}[t]\|$ and $\|\tilde{w}\|$ are continuous. Thus, considering (3.43), (3.48) and (3.23), it can be shown that

$$V_2(t) \leq e^{-2\upsilon_{etc}^{\star}(t-t_j)} V_2(t_j), \tag{3.49}$$

for all $t \in (t_j, t_{j+1}), j \in \mathbb{N}$. But we have that

$$\begin{aligned} V_2(t_j^-) =& V_1(t_j^-) + f(t_j^-)d^2(t_j^-) + m(t_j^-) \\ =& V_1(t_j^-) + \omega_0 d^2(t_j^-) = V_1(t_j) + \omega_0 d^2(t_j^-), \end{aligned} \tag{3.50}$$

as $f(t_j^-) = \omega_0$ (see Lemma 3) and $m(t_j^-) = 0$ (since events are triggered according to (3.1)). On the other hand, we have that

$$V_2(t_j) = V_1(t_j) + f(t_j)d^2(t_j) + m(t_j) = V_1(t_j) + m(t_j), \tag{3.51}$$

as $d(t_j) = 0$. Thus, if $m(t_j)$ is chosen as in (3.21), we have that

$$V_2(t_j^-) = V_2(t_j). \tag{3.52}$$

Therefore, using (3.49) and (3.52) recursively, we can show that

$$\begin{aligned} V_2(t) \le& e^{-2\upsilon^\star_{etc}(t-t_j)} V_2(t_j) = e^{-2\upsilon^\star_{etc}(t-t_j)} V_2(t_j^-) \\ \le& e^{-2\upsilon^\star_{etc}(t-t_j)} \times e^{-2\upsilon^\star_{etc}(t_j-t_{j-1})} V_2(t_{j-1}) \\ &\vdots \\ \le& e^{-2\upsilon^\star_{etc}(t-t_j)} \times \prod_{i=1}^{i=j} e^{-2\upsilon^\star_{etc}(t_i-t_{i-1})} V_2(0) \\ =& e^{-2\upsilon^\star_{etc} t} V_2(0), \end{aligned} \tag{3.53}$$

for all $t > 0$. But we have that

$$V_2(0) = V_1(0) + f(0)d^2(0) + m(0) = V_1(0), \tag{3.54}$$

as $d(0) = 0$ and $m(0) = 0$, and

$$V_1(t) \le V_2(t), \tag{3.55}$$

for all $t \ge 0$. Thus, it follows from (3.53) that

$$V_1(t) \le e^{-2\upsilon^\star_{etc} t} V_1(0), \tag{3.56}$$

for all $t > 0$. Then, considering (2.8), (3.24), and the bounded invertibility of the transformations (2.15), (2.19), (2.25), (2.31), we can obtain that

$$\|u[t]\| + \|\hat{u}[t]\| \le M_{etc} e^{-\upsilon^\star_{etc} t} \big(\|u[0]\| + \|\hat{u}[0]\| \big), \tag{3.57}$$

for all $t > 0$ for some $M_{etc} > 0$. This completes the proof. □

REMARK 2. The Lyapunov function $V_2(t)$ for the closed-loop system, as defined in (3.23), is carefully designed to ensure dissipation, even when the system evolves in open-loop between events. To counteract the effects of control input sampling, the functions $f(t)d^2(t)$ and $m(t)$ play crucial roles, each influencing different time intervals. Specifically, during $t \in (t_j, t_j + \tau), j \in \mathbb{N}$, the function $f(t)$, which is strictly decreasing and modulated by $d^2(t)$, helps to mitigate the impact of input sampling. In contrast, during $t \in (t_j + \tau, t_{j+1}), j \in \mathbb{N}$, the dynamic variable $m(t)$ assumes this role, ensuring dissipation throughout the process.

4. Periodic event-triggered control

In this section, we introduce a triggering mechanism that updates the control input only at specific events, requiring only periodic evaluations of the triggering function to detect these events. This approach is referred to as *periodic event-triggering*. We formulate a suitable triggering function, $\tilde{m}(t)$, and derive an upper bound for the sampling period h used in these periodic evaluations. By periodically assessing $\tilde{m}(t)$ and updating the control input whenever $\tilde{m}(t) < 0$ at an evaluation instant $t = nh$, $n \in \mathbb{N}$, it is ensured that the dynamic variable $m(t)$, evolving according to (3.2), (3.3) along the PETC trajectory, satisfies $m(t) \geq 0$ for all $t > 0$. Consequently, GES of the PETC closed-loop system—analogous to the GES established under CETC in Theorem 1—can be guaranteed. The detailed design is presented below.

DEFINITION 2. Let $\eta, \beta_1, \beta_2, \beta_3 > 0$ be free event-trigger parameters, and let $\rho > 0$ be an event-trigger parameter to be determined later. The set of PETC event times $I = \{t_j\}_{j\in\mathbb{N}}$, is defined as an increasing sequence satisfying $\lim_{j\to\infty} t_j = +\infty$ according to the rule:

$$t_{j+1} = \inf\left\{t \geq t_j + \tau \,\middle|\, \tilde{m}(t) < 0, t = nh, h > 0, n \in \mathbb{N}\right\}, \tag{4.1}$$

where $\tau > 0$ is the MDT to be chosen, $h > 0$ is a sampling period to be determined, and $\tilde{m}(t)$ is the periodic event-triggering function defined as

$$\tilde{m}(t) := m(t) - \frac{\rho}{a}\left(e^{ah} - 1\right)d^2(t), \tag{4.2}$$

where $a > 0$ is given by

$$a = 1 + \rho_1 + \eta, \tag{4.3}$$

with ρ_1 defined in (3.6), $d(t)$ is given by (2.30), and $m(t)$ satisfies (3.2) for all $t \in (t_j, t_j + \tau), j \in \mathbb{N}$, and (3.3) for all $t \in (t_j + \tau, t_{j+1}), j \in \mathbb{N}$, with the initial condition $m(t_0) = m(0) = 0$. Moreover, at each event time, $m(t_j) \geq 0$ for $j \in \mathbb{N}_{>0}$ is to be appropriately selected. At $t = t_j + \tau$, it is chosen that $m\big((t_j + \tau)^-\big) = m(t_j + \tau), j \in \mathbb{N}$, ensuring continuity.

Prior to presenting the main results under PETC, we first establish several auxiliary results stated in Lemmas 4 and 5. These results play a crucial role in proving the main findings outlined in Theorem 2.

LEMMA 4. Consider the PETC approach (2.21)–(2.24), (4.1)–(4.3), which generates an increasing sequence of event times $I = \{t_j\}_{j\in\mathbb{N}}$ with $t_0 = 0$, satisfying $\lim_{j\to\infty} t_j = +\infty$. Also, consider the dynamic variable $m(t)$, governed by (3.2), (3.3), along the solution of (2.1)–(2.3), (2.5)–(2.14), (2.21)–(2.24), (4.1)–(4.3), with initial condition $m(t_0) = 0$, with $m(t_0) = 0$, $m(t_j) \geq 0, j \in \mathbb{N}_{>0}$ to be chosen, and $m\big((t_j+\tau)^-\big) = m(t_j+\tau), j \in \mathbb{N}$, where $\tau > 0$ is the MDT to be chosen. Further, let the sampling period $h > 0$ be chosen such that

$$0 < h \le \min\{\tau, \tau_1, \tau_2, \tau_3\}, \text{ and } \tau/h \in \mathbb{N}, \tag{4.4}$$

where

$$\tau_i := \frac{1}{a}\ln\left(1 + \frac{\beta_i a}{\alpha_i \rho}\right), \quad i = 1, 2, 3, \tag{4.5}$$

with a given by (4.3) and for any $\beta_1, \beta_2, \beta_3, \eta, \rho > 0$, and $\alpha_1, \alpha_2, \alpha_3 > 0$ given by (3.7)–(3.9). Then, it follows that

$$m(t) \ge \frac{\rho}{a}\left(\frac{a}{\rho}m(nh) - \left(e^{a(t-nh)} - 1\right)d^2(nh)\right)e^{-\eta(t-nh)}, \tag{4.6}$$

for all $t \in \big[nh, (n+1)h\big)$ and $n \in \big[(t_j+\tau)/h, t_{j+1}/h\big) \cap \mathbb{N}$. In (4.6), $d(t)$ is given by (2.30).

Proof. If $m(t_0) = 0$, $m(t_j) \ge 0, j \in \mathbb{N}_{>0}$ to be chosen, and $m\big((t_j+\tau)^-\big) = m(t_j+\tau), j \in \mathbb{N}$, it follows from arguments similar to those in Lemma 1 that $m(t) \ge 0$ for all $t \in (t_j, t_j+\tau], j \in \mathbb{N}$.

For $t \in (nh, (n+1)h)$, $n \in \Big[(t_j+\tau)/h, t_{j+1}/h\Big) \cap \mathbb{N}$, applying Young's inequality and incorporating (3.5), we obtain:

$$\begin{aligned}\frac{d}{dt}d^2(t) &= 2d(t)\dot{d}(t) \le d^2(t) + \dot{d}^2(t) \\ &\le (1+\rho_1)d^2(t) + \alpha_1\|\hat{u}[t]\|^2 + \alpha_2\hat{u}^2(1,t) + \alpha_3\tilde{u}^2(0,t).\end{aligned} \tag{4.7}$$

Since both sides of (4.7) are well-defined, there exists a function $\iota(t) \ge 0$ such that:

$$\frac{d}{dt}d^2(t) = (1+\rho_1)d^2(t) + \alpha_1\|\hat{u}[t]\|^2 + \alpha_2\hat{u}^2(1,t) + \alpha_3\tilde{u}^2(0,t) - \iota(t), \tag{4.8}$$

for all $t \in (nh, (n+1)h)$ and $n \in \Big[(t_j+\tau)/h, t_{j+1}/h\Big) \cap \mathbb{N}$.

Combining (3.3) with (4.8), we obtain the system:

$$\dot{z}(t) = \mathscr{A}z(t) + \psi(t), \tag{4.9}$$

where:

$$\begin{aligned}z(t) &= \begin{bmatrix} m(t) \\ d^2(t)\end{bmatrix}, \quad \mathscr{A} = \begin{bmatrix} -\eta & -\rho \\ 0 & 1+\rho_1\end{bmatrix}, \\ \psi(t) &= \begin{bmatrix} \beta_1\|\hat{u}[t]\|^2 + \beta_2\hat{u}^2(1,t) + \beta_3\tilde{u}^2(0,t) \\ \alpha_1\|\hat{u}[t]\|^2 + \alpha_2\hat{u}^2(1,t) + \alpha_3\tilde{u}^2(0,t) - \iota(t)\end{bmatrix}.\end{aligned} \tag{4.10}$$

The solution for (4.9) is given by:

$$z(t) = e^{\mathscr{A}(t-nh)}z(nh) + \int_{nh}^{t} e^{\mathscr{A}(t-\xi)}\psi(\xi)d\xi, \tag{4.11}$$

for all $t \in [nh, (n+1)h)$ and $n \in \Big[(t_j + \tau)/h, t_{j+1}/h\Big) \cap \mathbb{N}$. Consequently,

$$m(t) = Ce^{\mathscr{A}(t-nh)}z(nh) + \int_{nh}^{t} Ce^{\mathscr{A}(t-\xi)}\psi(\xi)d\xi, \tag{4.12}$$

where

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}. \tag{4.13}$$

The eigenvalues of $\mathscr{A}$ are $-\eta$ and $1+\rho_1$. Using matrix diagonalization, we express $e^{\mathscr{A}t}$ as:

$$e^{\mathscr{A}t} = \begin{bmatrix} 1 & -\frac{\rho}{a} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^{-\eta t} & 0 \\ 0 & e^{(1+\rho_1)t} \end{bmatrix} \begin{bmatrix} 1 & \frac{\rho}{a} \\ 0 & 1 \end{bmatrix}, \tag{4.14}$$

where a is given by (4.3). Therefore,

$$\begin{aligned} Ce^{\mathscr{A}(t-\xi)}\psi(\xi) =& \Big(\beta_1 g_1(t-\xi) - \alpha_1 g_2(t-\xi)\Big)\|\hat{u}[\xi]\|^2 \\ &+ \Big(\beta_2 g_1(t-\xi) - \alpha_2 g_2(t-\xi)\Big)\hat{u}^2(1,t) \\ &+ \Big(\beta_3 g_1(t-\xi) - \alpha_3 g_2(t-\xi)\Big)\tilde{u}^2(0,t) \\ &+ g_2(t-\xi)\iota(\xi), \end{aligned} \tag{4.15}$$

where

$$g_1(t) = e^{-\eta t} \geq 0, \quad g_2(t) = \frac{\rho}{a}(e^{at} - 1)e^{-\eta t} \geq 0. \tag{4.16}$$

Considering (4.16), recalling that $nh \leq \xi \leq t < (n+1)h$, and that $h > 0$ has been chosen as in (4.4)–(4.3), it can be shown that

$$\begin{aligned} &\beta_i g_1(t-\xi) - \alpha_i g_2(t-\xi) \\ &= \frac{\alpha_i \rho}{a}\left(1 + \frac{\beta_i a}{\alpha_i \rho} - e^{a(t-\xi)}\right)e^{-\eta(t-\xi)} \\ &\geq \frac{\alpha_i \rho}{a}\left(1 + \frac{\beta_i a}{\alpha_i \rho} - e^{ah}\right)e^{-\eta h} \\ &\geq 0, \end{aligned} \tag{4.17}$$

for $i = 1, 2, 3$. Thus, it follows from (4.15) that

$$Ce^{\mathscr{A}(t-\xi)}\psi(\xi) \geq 0, \tag{4.18}$$

for all t, ξ such that $nh \leq \xi \leq t < (n+1)h$, and $n \in \big[(t_j + \tau)/h, t_{j+1}/h\big) \cap \mathbb{N}$. Therefore, considering (4.12) and (4.18), it can be shown that

$$\begin{aligned} m(t) &\geq Ce^{\mathscr{A}(t-nh)}z(nh) \\ &= \frac{\rho}{a}\left(\frac{a}{\rho}m(nh) - \big(e^{a(t-nh)} - 1\big)d^2(nh)\right)e^{-\eta(t-nh)}. \end{aligned} \tag{4.19}$$

for all $t \in \big[nh, (n+1)h\big)$ and $n \in \big[(t_j + \tau)/h, t_{j+1}/h\big) \cap \mathbb{N}$. This concludes the proof. □

Lemma 5. Consider the PETC approach (2.21)–(2.24), (4.1)–(4.3) which generates a set of increasing event times $I = \{t_j\}_{j\in\mathbb{N}}$ with $t_0 = 0$, satisfying $\lim_{j\to\infty} t_j = +\infty$. Also consider the dynamic variable $m(t)$ governed by (3.2), (3.3) along the solution of (2.1)–(2.3), (2.5)–(2.14), (2.21)–(2.24), (4.1)–(4.3) with $m(t_0) = 0$, $m(t_j) \geq 0, j \in \mathbb{N}_{>0}$ to be chosen, and $m\big((t_j + \tau)^-\big) = m(t_j + \tau), j \in \mathbb{N}$, where $\tau > 0$ is the MDT to be chosen. Let the sampling period $h > 0$ be chosen as in (4.4), (4.5). Then, it holds that $m(t) \geq 0$ for all $t > 0$.

Proof. Suppose an event is triggered at time $t = t_j$ and that $m(t_j) \geq 0, j \in \mathbb{N}$. From Lemma 4, it follows that $m(t)$ remains non-negative over the interval $(t_j, t_j + \tau]$, for all $j \in \mathbb{N}$. The periodic event-triggering mechanism, defined in (4.1)–(4.5), is evaluated at discrete instants $t = nh$. An event is triggered only if $t \geq t_j + \tau$ and $\tilde{m}(nh) < 0$, which corresponds to the condition

$$m(nh) < \frac{\rho}{a}\big(e^{ah} - 1\big)d^2(nh), \tag{4.20}$$

indicating that a control update is required. Conversely, when $t \geq t_j + \tau$ and $\tilde{m}(nh) \geq 0$, that is,

$$m(nh) \geq \frac{\rho}{a}\big(e^{ah} - 1\big)d^2(nh), \tag{4.21}$$

no update is necessary, as it can be ensured that $m(t) \geq 0$ for all $t \in [nh, (n+1)h)$. This follows from the fact that the right-hand side of (4.6) remains non-negative for all $t \in [nh, (n+1)h)$ when $m(nh)$ satisfies the stated inequality. Consequently, $m(t)$ remains non-negative at least until $t = t_{j+1}$, where $\tilde{m}(t_{j+1}^-) < 0$, i.e.

$$m(t_{j+1}^-) < \frac{\rho}{a}\big(e^{ah} - 1\big)d^2(t_{j+1}^-). \tag{4.22}$$

At this point, a control should be updated, and $m(t_{j+1}) > 0$ is selected. Thus, recalling that the initial conditions are chosen as $m(t_0) = 0$ and $m(t_j) \geq 0$ for all $j \in \mathbb{N}$, it follows that $m(t) \geq 0$ for all $t \in (t_j + \tau, t_{j+1})$, for every $j \in \mathbb{N}$. Since the sequence of event times does not lead to Zeno, this implies $m(t) \geq 0$ for all $t > 0$, completing the proof. □

THEOREM 2. Consider the PETC approach (2.21)–(2.24), (4.1)–(4.3), which generates a set of increasing event times $\{t_j\}_{j\in\mathbb{N}}$ with $t_0 = 0$, satisfying $\lim_{j\to\infty} t_j = +\infty$. For every $u[0], \hat{u}[0] \in L^2(0,1)$, there exist unique solutions $u, \hat{u} : \mathbb{R}_+ \times [0,1] \to \mathbb{R}$ such that $u, \hat{u} \in C^0(\mathbb{R}_+; L^2(0,1) \cap C^1(J \times [0,1])$ with $u[t] \in C^2([0,1])$ which satisfy (2.2), (2.3), (2.6), (2.7), (2.13), (2.14), (2.21)–(2.24) for all $t > 0$ and (2.1), (2.5), (2.12), (2.14) for all $t > 0, x \in (0,1)$, where $J = \mathbb{R}_+ \backslash I$. Let the event-trigger parameters $\beta_1, \beta_2, \beta_3, \eta > 0$ be free parameters. Further, let $a_1, a_2 > 0, \omega_1 > \omega_0 > 0$, and subject to Assumption 1, let $a_0 > 0$ be chosen as in (3.14), and the event-trigger parameter $\rho > 0$ be chosen as in (3.18). Let the MDT $\tau > 0$ be chosen as in (3.11), the sampling time $h > 0$ be chosen as in (4.4), (4.5) and the initial conditions of $m(t)$ governed by (3.2), (3.3) along the solution of (2.1)–(2.3), (2.5)–(2.14), (2.21)–(2.24), (4.1) be chosen as in (3.19)–(3.21). Then, under the PETC approach (2.21)–(2.24), (4.1)–(4.3), the closed-loop system consisting of the plant (2.1)–(2.3) and the observer (2.5)–(2.14) is globally exponentially stable, satisfying the estimate (3.22).

Proof. The well-posedness, as stated in Theorem 1, is established by repeatedly applying Proposition 1 between successive triggering events. Given that the initial conditions of $m(t)$ are set according to (3.19)–(3.21), Lemma 5 ensures that $m(t)$ remains non-negative for all $t > 0$. Consequently, by following the same reasoning as in the proof of Theorem 1, we can conclude that the closed-loop system under the PETC strategy exhibits GES satisfying the estimate (3.22). □

REMARK 3. Theorem 1 establishes that in the CETC framework, the parameters $\beta_1, \beta_2, \beta_3$ must satisfy the conditions $\beta_1, \beta_2, \beta_3 \geq 0$ with at least one being strictly positive, i.e., $\beta_1 + \beta_2 + \beta_3 > 0$. However, in the PETC setting, as outlined in Lemma 4 and Theorem 2, these parameters are required to be strictly positive. This guarantees the existence of a constant $h > 0$ ensuring that the relation (4.6) holds for all $t \in \big[nh, (n+1)h\big)$ and $n \in \big[(t_j + \tau)/h, t_{j+1}/h\big) \cap \mathbb{N}$.

REMARK 4. While the sampling period $h > 0$ must satisfy $h < \tau$ where $\tau > 0$ denotes the MDT, its value can be chosen with some flexibility. In the special case $\Delta = 0$, from (3.12) we find

$$\tau = \frac{\omega_1 - \omega_0}{a_2\left(\omega_1 + \frac{a_1}{2a_2}\right)\left(\omega_0 + \frac{a_1}{2a_2}\right)}. \tag{4.23}$$

Rather than computing τ exactly, we may select any $\tau > 0$ satisfying

$$0 < \tau < \frac{1}{a_2\left(\omega_0 + \frac{a_1}{2a_2}\right)}, \tag{4.24}$$

and then determine $\omega_1 > \omega_0$ from (4.23) as

$$\omega_1 = \frac{\omega_0 + \frac{\tau a_1}{2}\left(\omega_0 + \frac{a_1}{2a_2}\right)}{1 - \tau a_2\left(\omega_0 + \frac{a_1}{2a_2}\right)}. \tag{4.25}$$

Hence, the MDT $\tau > 0$ and, consequently, the sampling period $h > 0$ can be chosen with a certain degree of freedom.

5. Self-triggered control

In this section, we focus solely on the problem of full-state feedback control, disregarding any observer-induced effects discussed in Section 2. Consequently, the full-state feedback sampled-data control input is given by

$$U_j = \int_0^1 k(y)u(y, t_j)dy, \tag{5.1}$$

for all $t \in [t_j, t_{j+1})$, where $j \in \mathbb{N}$ and $k(y)$ is specified by (2.22)–(2.24). The input holding error due to sampling, denoted by $d(t)$, is expressed as

$$d(t) = \int_0^1 k(y)\big(u(y, t_j) - u(y, t)\big)dy, \tag{5.2}$$

for all $t \in [t_j, t_{j+1})$ and $j \in \mathbb{N}$.

Unlike CETC and PETC, the proposed STC does not monitor any triggering function to decide when to update the control input. Instead, the next event time t_{j+1} is determined directly at the current event time t_j, utilizing only the measurements obtained at $t = t_j$ and $t = t_{j-1}$. In contrast, CETC and PETC require *continuous* measurements from the plant to compute the dynamic variable $m(t)$ used in the triggering function.

Definition 3. Let $\eta, \omega_0 > 0$ be free event-trigger parameters, and let $\rho > 0$ be an event-trigger parameter to be determined later. The set of STC event times $I = \{t_j\}_{j\in\mathbb{N}}$, is defined as an increasing sequence satisfying $\lim_{j\to\infty} t_j = +\infty$ according to the rule:

$$\begin{aligned} t_1 =& t_0 + \tau, \\ t_{j+1} =& t_j + \tau + \frac{1}{2\lambda + \eta} \ln\left(1 + \frac{(2\lambda + \eta)G\big(u[t_{j-1}], u[t_j]\big)}{\rho e^{(2\lambda+\eta)\tau}(H(u[t_j]) + \epsilon_0)}\right), j \in \mathbb{N}_{>0}, \end{aligned} \tag{5.3}$$

where $\tau > 0$ is the MDT to be set, $\epsilon_0 > 0$,

$$\begin{aligned} G\big(u[t_{j-1}], u[t_j]\big) :=& \omega_0 d^2(t_j^-) \\ =& \omega_0\Big(\int_0^1 k(y)\big(u(y, t_{j-1}) - u(y, t_j)\big)dy\Big)^2, \end{aligned} \tag{5.5}$$

$$H(u[t_j]) := 2\|k\|^2\Big(2 + \frac{\varepsilon^2\|k\|^2}{\lambda^2}\Big)\|u[t_j]\|^2, \tag{5.5}$$

with $k(y)$ given by (2.22)–(2.24).

Prior to presenting the main results under STC, we first establish several auxiliary results stated in Lemmas 6 and 7. These results play a crucial role in proving the main findings outlined in Theorem 3.

Lemma 6. Consider the STC approach (5.1), (5.3)–(5.5), which generates an increasing sequence of event times $\{t_j\}_{j\in\mathbb{N}}$ with $t_0 = 0$, satisfying $\lim_{j\to\infty} t_j = +\infty$. Then, for the closed-loop system (2.1)–

(2.3), (5.1), and the error $d(t)$ defined in (5.2), the following estimates hold:

$$\|u[t]\|^2 \leq \left(1 + \frac{\varepsilon^2\|k\|^2}{\lambda^2}\right)\|u[t_j]\|^2 e^{2\lambda(t-t_j)}, \tag{5.6}$$

and

$$d^2(t) \leq H(u[t_j])e^{2\lambda(t-t_j)}, \tag{5.7}$$

for all $t \in (t_j, t_{j+1}), j \in \mathbb{N}$, where $k(y)$ is given by (2.22)–(2.24), and $H(u[t_j])$ is given by (5.5).

The proof is very similar to that of Lemma 3 of Rathnayake & Diagne (2024b), and hence is omitted.

Lemma 7. Consider the STC approach in (5.1), (5.3)–(5.5), which generates an increasing sequence of event times $\{t_j\}_{j\in\mathbb{N}}$ with $t_0 = 0$, satisfying $\lim_{j\to\infty} t_j = +\infty$. Then, the dynamic variable $m(t)$ governed by

$$\dot{m}(t) = -\eta m(t) + \beta_1\|u[t]\|^2 + \beta_2 u^2(1,t), \tag{5.8}$$

for all $t \in (t_j, t_j + \tau), j \in \mathbb{N}$, and

$$\dot{m}(t) = -\eta m(t) - \rho d^2(t) + \beta_1\|u[t]\|^2 + \beta_2 u^2(1,t), \tag{5.9}$$

for all $t \in [t_j + \tau, t_{j+1}), j \in \mathbb{N}$, with $\tau > 0$ denoting the MDT to be selected, and constants $\eta, \rho > 0$, $\beta_1, \beta_2 \geq 0$, along with $m(t_0) = m(0) = 0$, $m((t_j+\tau)^-) = m(t_j+\tau)$, and $m(t_j)$ for $j \in \mathbb{N}_{>0}$ defined as $m(t_j) = G(u[t_{j-1}], u[t_j])$ for any $G(u[t_{j-1}], u[t_j]) \geq 0$, satisfies $m(t) \geq 0$ for all $t > 0$.

Proof. Considering (5.8), we obtain

$$\begin{aligned} m(t) &= e^{-\eta(t-t_j)}m(t_j) + \int_{t_j}^{t} e^{-\eta(t-\xi)}\left(\beta_1\|u[\xi]\|^2 + \beta_2 u^2(1,\xi)\right)d\xi \\ &\geq e^{-\eta(t-t_j)}m(t_j), \end{aligned} \tag{5.10}$$

for all $t \in (t_j, t_j + \tau], j \in \mathbb{N}$. If $m(t_j)$ is chosen as $m(0) = 0$ or $m(t_j) = G(u[t_{j-1}], u[t_j])$, for any $G(u[t_{j-1}], u[t_j]) \geq 0$, it follows that $m(t) \geq 0$ for all $t \in (t_j, t_j + \tau], j \in \mathbb{N}$.

Now, consider the time period when $t \in (t_j + \tau, t_{j+1}), j \in \mathbb{N}_{>0}$. Recalling (5.7) and using (5.9), we obtain

$$\dot{m}(t) \geq -\eta m(t) - \rho H(u[t_j])e^{2\lambda(t-t_j)}, \tag{5.11}$$

from which it follows that

$$m(t) \geq m(t_j+\tau)e^{-\eta(t-t_j-\tau)} - \frac{\rho H(u[t_j])e^{2\lambda\tau}}{2\lambda+\eta}e^{-\eta(t-t_j-\tau)}\left(e^{(2\lambda+\eta)(t-t_j-\tau)} - 1\right), \tag{5.12}$$

for all $t \in [t_j + \tau, t_{j+1}), j \in \mathbb{N}_{>0}$. Using (5.10), we note that $m(t_j + \tau) \geq e^{-\eta\tau} m(t_j)$, so we obtain

$$m(t) \geq m(t_j)e^{-\eta\tau} e^{-\eta(t-t_j-\tau)} - \frac{\rho H(u[t_j])e^{2\lambda\tau}}{2\lambda + \eta} e^{-\eta(t-t_j-\tau)} \left(e^{(2\lambda+\eta)(t-t_j-\tau)} - 1\right), \tag{5.13}$$

for all $t \in [t_j + \tau, t_{j+1}), j \in \mathbb{N}_{>0}$.

Define the function $c_j : \mathbb{R}_{\geq 0} \to \mathbb{R}$ as follows:

$$c_j(s) := m(t_j)e^{-\eta\tau} + \frac{\rho H(u[t_j])e^{2\lambda\tau}}{2\lambda + \eta} - \frac{\rho H(u[t_j])e^{2\lambda\tau}}{2\lambda + \eta} e^{(2\lambda+\eta)s}. \tag{5.14}$$

We observe that

$$c_j(0) = m(t_j)e^{-\eta\tau} \geq 0, \tag{5.15}$$

and

$$c_j'(s) = -\rho H(u[t_j])e^{2\lambda\tau} e^{(2\lambda+\eta)s} \leq 0, \tag{5.16}$$

which implies that $c_j(s)$ is nonincreasing for all $s \geq 0$. Therefore, if $m(t_j) = G(u[t_{j-1}], u[t_j])$ with $G(u[t_{j-1}], u[t_j]) > 0$ and $H(u[t_j]) > 0$, i.e., $\|u[t_j]\| > 0$, then there exists a unique $s^\star > 0$ given by

$$s^\star = \frac{1}{2\lambda + \eta} \ln\left(1 + \frac{(2\lambda + \eta)m(t_j)}{\rho e^{(2\lambda+\eta)\tau} H(u[t_j])}\right), \tag{5.17}$$

such that

$$c_j(s^\star) = 0, \tag{5.18}$$

and

$$c_j(0) > c_j(s) > 0, \tag{5.19}$$

for all $s \in (0, s^\star)$. Therefore, if events are triggered according to (5.3), where any $\varepsilon_0 > 0$, it definitely follows from (5.13) that $m(t) \geq 0$ for all $t \in (t_j + \tau, t_{j+1}), j \in \mathbb{N}_{>0}$.

Thus, since Zeno behaviour inherently cannot occur under the STC approach (5.1), (5.3)–(5.5), we conclude that $m(t) \geq 0$ for all $t > 0$. □

THEOREM 3. Consider the STC approach (5.1), (5.3)–(5.5), which generates a set of increasing event times $I = \{t_j\}_{j\in\mathbb{N}}$ with $t_0 = 0$, satisfying $\lim_{j\to\infty} t_j = +\infty$. For every $u[0] \in L^2(0, 1)$, there exists a unique solution $u : \mathbb{R}_+ \times [0, 1] \to \mathbb{R}$ such that $u \in C^0(\mathbb{R}_+; L^2(0, 1)) \cap C^1(J \times [0, 1])$ with $u[t] \in C^2([0, 1])$, satisfying (2.2), (2.3), (5.1) for all $t > 0$, and (2.1) for all $t > 0$, $x \in (0, 1)$, where $J = \mathbb{R}_+ \setminus I$. Let

$a_1, a_2 > 0$, $\omega_1 > \omega_0 > 0$, and let $a_0 > 0$ be chosen such that

$$a_0 = \frac{\varepsilon \kappa_1 B}{2} + \frac{3\varepsilon^2 k^2(1)}{a_2}, \tag{5.20}$$

where $k(y)$ satisfies (2.22), and subject to Assumption 1, the parameters $B, \kappa_1 > 0$ are chosen such that

$$B\left(\varepsilon \min\left\{r - \frac{1}{2}, \frac{1}{2}\right\} - \frac{\varepsilon}{2\kappa_1}\right) - 2\beta_1 \tilde{L}^2 - 4\beta_2 \check{L}^2 - 2\beta_2 - \frac{2\alpha_1 \tilde{L}^2 + 4\alpha_2 \check{L}^2 + 2\alpha_2}{a_2} > 0, \tag{5.21}$$

with $\alpha_1, \alpha_2 > 0$ given by

$$\alpha_1 = 3\int_0^1 \left(\varepsilon k''(y) + \varepsilon k(1)k(y) + \lambda k(y)\right)^2 dy, \tag{5.22}$$

$$\alpha_2 = 3\left(\varepsilon q k(1) + \varepsilon k'(1)\right)^2, \tag{5.23}$$

and $\beta_1, \beta_2 \geq 0$ satisfying $\beta_1 + \beta_2 > 0$. Let the event-trigger parameter $\rho > 0$ be chosen as

$$\rho = a_2 \omega_0^2 + a_1 \omega_0 + a_0, \tag{5.24}$$

and let $\eta > 0$ be a free event-trigger parameter. Furthermore, let the MDT $\tau > 0$ be chosen as in (3.11). Then, under the STC approach (5.1), (5.3)–(5.5), the closed-loop system (2.1)–(2.3) is globally exponentially stable. More specifically, the following estimate holds:

$$\|u[t]\| \leq M_{stc} e^{-\upsilon^{\star}_{stc} t} \|u[0]\|, \tag{5.25}$$

where $M_{stc} > 0$ and $\upsilon^{\star}_{stc} > 0$ are some positive constants.

Proof. The well-posedness in the sense of Theorem 3 is obtained by iteratively applying the results of Proposition 1 pertaining to the plant (2.1)–(2.3) in between two events.

Let the initial conditions of $m(t)$ governed by (5.8), (5.9) be chosen as

$$m(t_0) = m(0) = 0, \tag{5.26}$$

$$m\big((t_j + \tau)^-\big) = m(t_j + \tau), \text{ for } j \in \mathbb{N}, \tag{5.27}$$

$$m(t_j) = \omega_0 d^2(t_j^-), \text{ for } j \in \mathbb{N}_{>0}, \omega_0 > 0, \tag{5.28}$$

where $d(t)$ is given by (5.2). Then, note from Lemma 7 that $m(t)$ satisfies $m(t) \geq 0$ for all $t > 0$. Let us consider a Lyapunov candidate defined as

$$V_3(t) := \frac{B}{2}\|w[t]\|^2 + f(t)d^2(t) + m(t), \tag{5.29}$$

where $f(t) : \mathbb{R}_{\geq 0} \to \mathbb{R} > 0$ satisfies (3.10), and B is chosen to satisfy (5.21). Note from Lemma 3 that $f(t)$ remains non-negative when MDT $\tau > 0$ is chosen as (3.11). In (5.29), w is given by the transformation

$$w(x,t) = u(x,t) - \int_0^x K(x,y)u(y,t)dy, \tag{5.30}$$

where $K(x,y)$ is given by (2.24). The inverse transformation of (5.30) is

$$u(x,t) = w(x,t) + \int_0^x L(x,y)w(y,t)dy, \tag{5.31}$$

where $L(x,y)$ is given by (2.32). Application of (5.30) on (2.1)–(2.3) together with the STC input (5.1) leads to the target system

$$w_t(x,t) = \varepsilon w_{xx}(x,t), \forall x \in (0,1), \tag{5.32}$$

$$w_x(0,t) = 0, \tag{5.33}$$

$$w_x(1,t) = -rw(1,t) + d(t), \tag{5.34}$$

valid for all $t \in [t_j, t_{j+1}), j \in \mathbb{N}$, where r is given by (2.23). Then, we can follow very similar proof procedure as the one presented in the proof of Theorem 1 to show that the closed-loop system is globally exponentially stable. More specifically, it can be shown that

$$\|u[t]\| \leq \tilde{K}\tilde{L}e^{-\upsilon^\star_{stc}t}\|u[0]\|, \tag{5.35}$$

for all $t > 0$, where $\tilde{L}$ is given by (3.16) and $\tilde{K} = 1 + \left(\int_0^1\int_0^x K^2(x,y)dydx\right)^{1/2}$ with $K(x,y)$ given by (2.24), and

$$\upsilon^\star_{stc} = \min\left\{\frac{\upsilon_{stc}}{B}, \frac{a_1}{2}, \frac{\eta}{2}\right\}, \tag{5.36}$$

with

$$\upsilon_{stc} := \frac{\varepsilon B}{4} - \beta_1\tilde{L}^2 - 2\beta_2\check{L}^2 - \frac{(\alpha_1\tilde{L}^2 + 2\alpha_2\check{L}^2)}{a_2} > 0. \tag{5.37}$$

□

6. Numerical simulations

We consider a RD system with constant parameters $\varepsilon = 0.1, \lambda = 0.25, q = 2$. The initial condition is selected as $u[0] = 10x^2(x-1)^2$. For both CETC and PETC, we consider the observer-based problem, where the initial conditions for the observer are chosen as $\hat{u}[0] \equiv 0$.

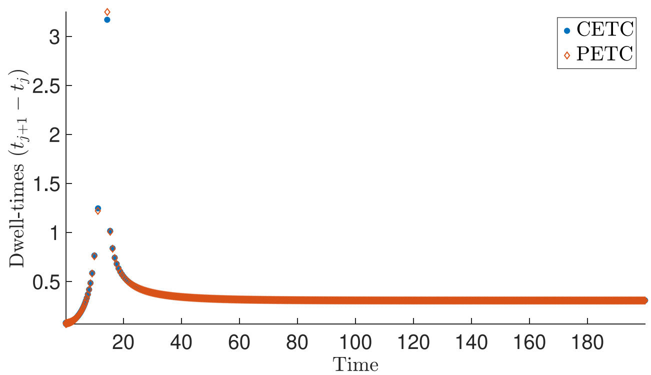

FIG. 1. Dwell-times under CETC and PETC.

6.1. *CETC and PETC*

The parameters for both the CETC and PETC triggering strategies are selected as follows: we set $\beta_i = 1$ for $i = 1, 2, 3$, along with $\eta = 1$, $a_2 = 1$, $\omega_0 = 10$ and $\omega_1 = 10000$. Based on (3.6)–(3.9), we determine the values $\rho_1 = 0.3525$, $\alpha_1 = 0.4622$, $\alpha_2 = 0.7207$ and $\alpha_3 = 3.9677$. We fix $\kappa_1 = 5$, and select $B = 1750.6$ to ensure that the condition in (3.15) is met. Utilizing (3.14), we compute $a_0 = 437.8159$, and then assign $a_1 = 2\sqrt{a_0 a_2} = 41.8481$. From equation (3.18), we obtain $\rho = 956.2968$. The resulting MDT is calculated to be 0.032. Consequently, we choose a time step of $\varDelta t = 0.0001$ for the numerical integration of the plant and observer dynamics. In line with equations (4.4) and (4.5), we set the PETC sampling interval as $h = 0.001$. For spatial discretization, a grid size of $\varDelta x = 0.005$ is adopted.

Figure 1 presents the dwell-times for CETC and PETC. The control inputs for both methods are shown in Fig. 2, demonstrating similar behaviour. Figure 3 shows the evolution of the L^2 norms under CETC and PETC, which remain closely aligned. Finally, Fig. 4 illustrates the L^2 norms of the observer error for both methods. The observer error is unaffected by the control input, since control does not appear in the system equations (2.9)–(2.11).

6.2. *STC*

The parameters for STC is selected as follows: we set $\beta_i = 1$ for $i = 1, 2$, along with $\eta = 1$, $a_2 = 1$, $\omega_0 = 10$, $\omega_1 = 10\,000$ and $\epsilon_0 = 10^{-6}$. Based on (5.22), (5.23), we determine the values $\alpha_1 = 0.3466$, $\alpha_2 = 0.5405$. We fix $\kappa_1 = 5$, and select $B = 1057.4$ to ensure that the condition in (5.21) is met. Utilizing (5.20), we compute $a_0 = 264.6262$, and then assign $a_1 = 2\sqrt{a_0 a_2} = 32.5347$. From equation (5.24), we obtain $\rho = 689.9729$. The resulting MDT is calculated to be 0.038. Consequently, we choose a time step of $\varDelta t = 0.0001$ for the numerical integration of the plant and observer dynamics. For spatial discretization, a grid size of $\varDelta x = 0.005$ is adopted.

Figure 5 shows the waiting time $t_{j+1} - t_j - \tau$ that satisfies (5.3)–(5.5), while Fig. 6 presents the control inputs under STC. Although events are triggered aperiodically, the behaviour appears nearly periodic with a period of τ. This is due to the conservative nature of STC, which does not check any triggering conditions and relies only on measurements sampled at event times. Figure 7 displays the evolution of the L^2 norms of the states under STC.

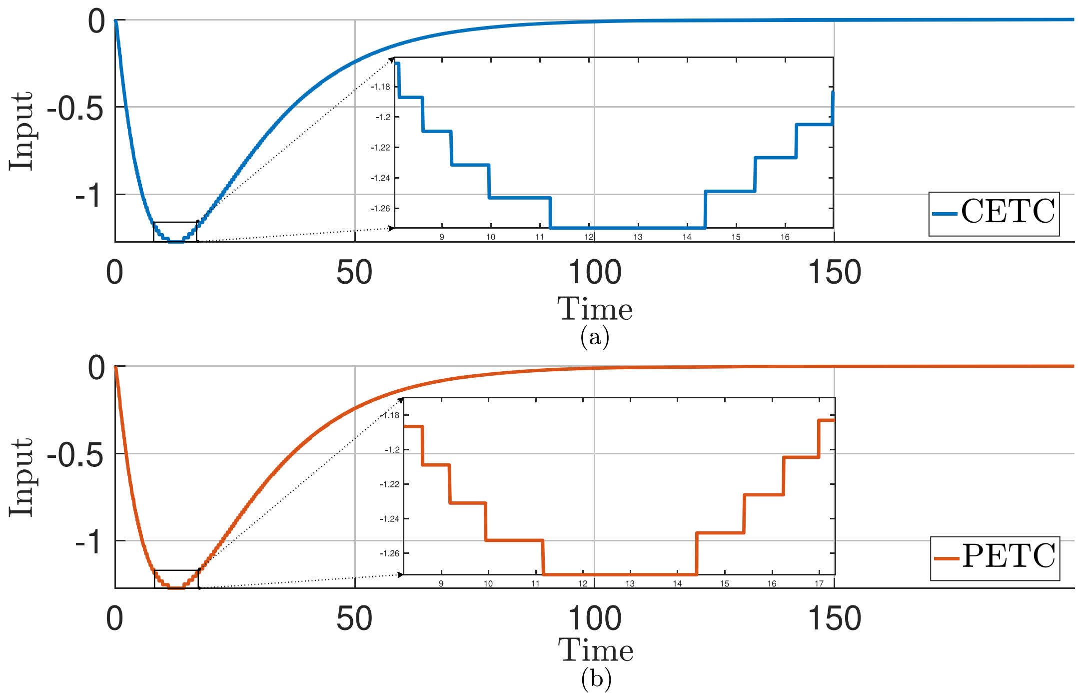

FIG. 2. CETC and PETC inputs.

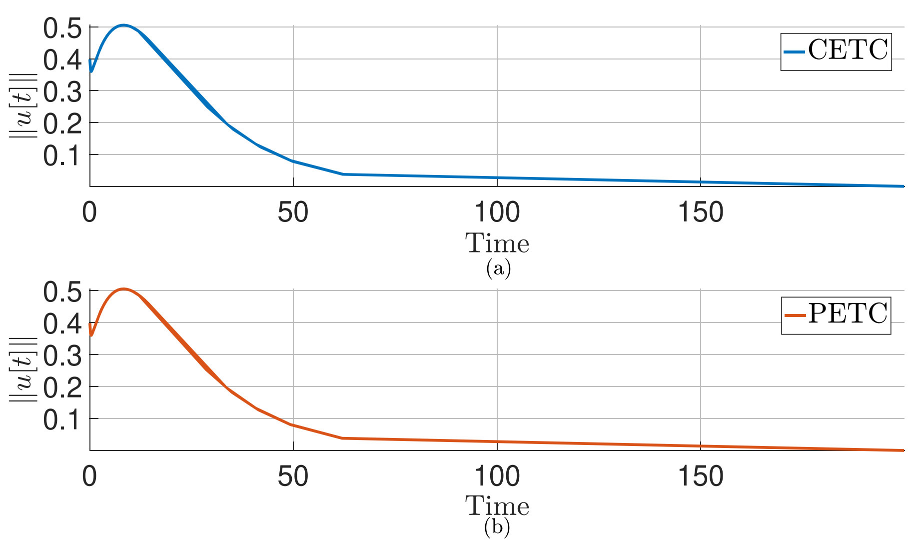

FIG. 3. Evolution of $\|u[t]\|$ under CETC and PETC Inputs.

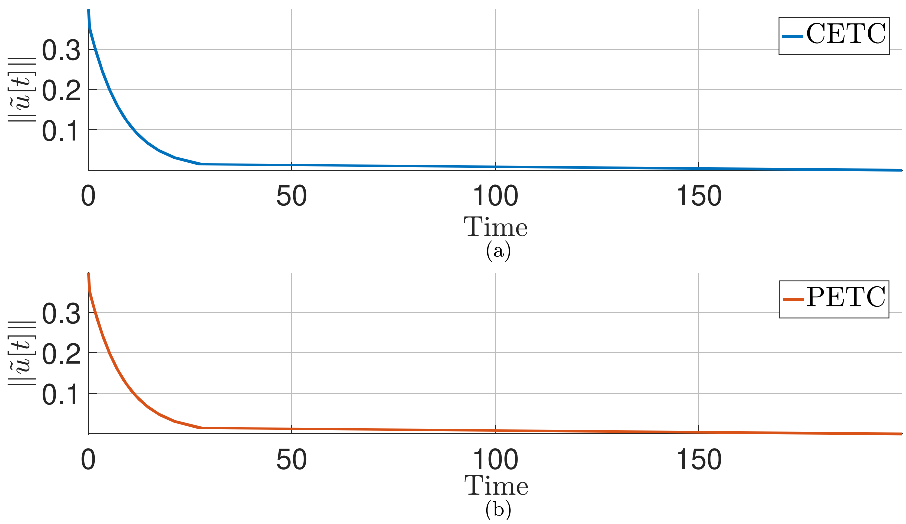

FIG. 4. Evolution of $\|\tilde{u}[t]\|$.

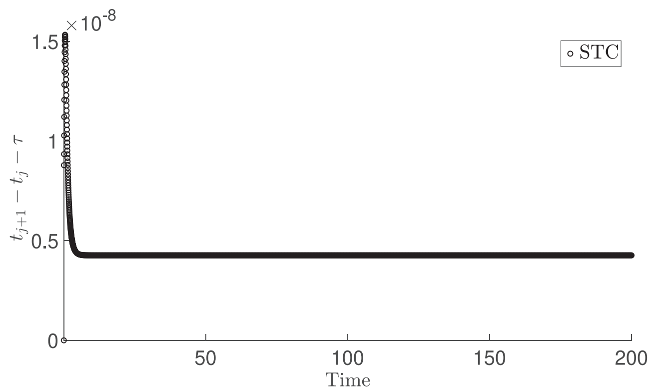

FIG. 5. Values of $t_{j+1} - t_j - \tau$ satisfying (5.3)–(5.5)

7. Conclusions

This paper presents three modular PDE backstepping-based ETC designs for a class of scalar one-dimensional RD systems. The proposed CETC, PETC and STC schemes achieve GES—a result not previously attained for RD PDEs when dynamic triggering was combined with PDE backstepping. The dynamic event generators are designed using time regularization, enabling the specification of a suitable MDT, along with a switching dynamic variable featuring a state-independent dynamic reset. A novel Lyapunov function is proposed to prove the GES claim. Moreover, our full-state feedback STC design eliminates the need for continuous monitoring of the event generator by determining the next triggering instant solely from measurements obtained at events. Simulation results are provided to support and illustrate the theoretical findings.

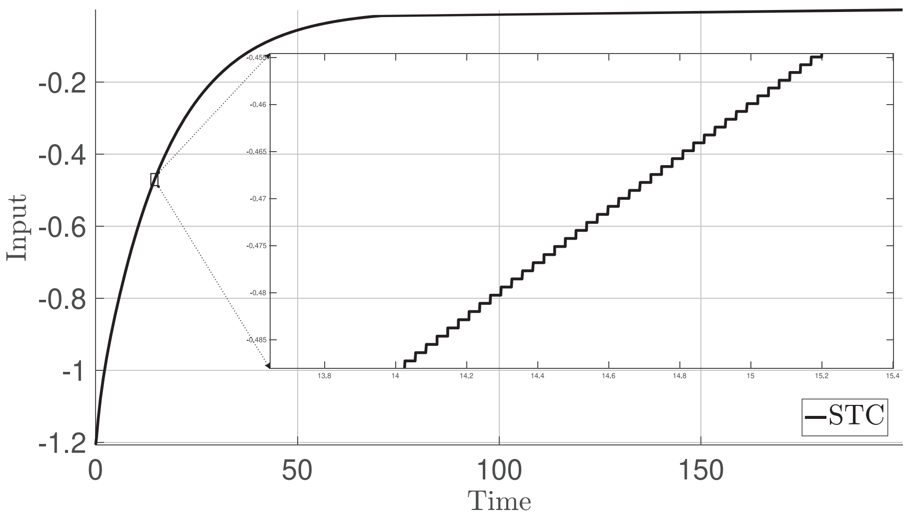

FIG. 6. STC input.

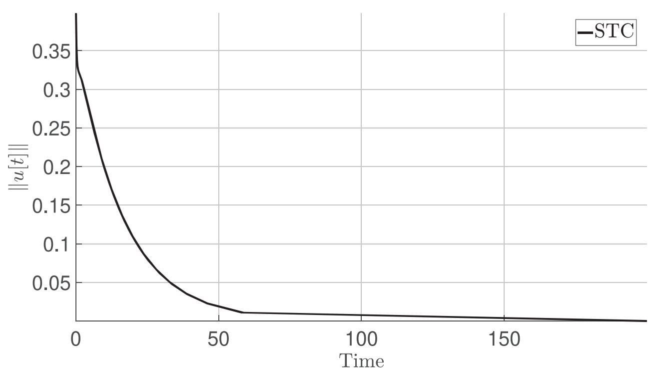

FIG. 7. Evolution of $\|u[t]\|$ under STC input.

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