

Feedback stabilization for the mass balance equations of an extrusion process

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Abstract—In this paper, we study the stabilization problem for a food extrusion process in the isothermal case. The model expresses the mass conservation in the extruder chamber and consists of a hyperbolic Partial Differential Equation (PDE) and a nonlinear Ordinary Differential Equation (ODE) whose dynamics describes the evolution of a moving interface. By using a Lyapunov approach, we obtain the exponential stabilization for the closed-loop system under natural feedback controls through indirect measurements. Numerical simulations are also provided with a comparison between the proposed approach and linear PI feedback controller.

Index Terms—Feedback stabilization, hyperbolic system, moving interface, Lyapunov approach.

I. INTRODUCTION

SCREW extruders have become very popular for their ability to manufacture food and plastics products with desired shapes and properties. Due to the strong interaction between the mass, the energy and the momentum balances occurring in those processes, the design of efficient controllers still remains a hard task at the industrial level. So far, the control oriented model of extruders are issued from some black box model of limited operational validity. Following the objectives of the control, these models describe the extruder's temperature and flow rate at the die output or the pressure dynamics based on single input and single output or multiple input and multiple output system. Generally, extrusion processes are controlled using *PID* [9], [12], [17] or predictive controllers [6], [19] with oversimplified or empirical models. In [12], the volumetric expansion of the extrudate correlated to the die temperature and pressure and the specific mechanical energy is chosen as the key product quality to be controlled. The authors study the performance of the *PI* controller based on the regulation of the die pressure using feed rate as a manipulative variable and show that the response of an improperly tuned controller may be too sluggish on one hand, or too oscillatory on the other hand. First-order, second-order and Lead-lag Laplace transfer-function are exploited in [18] to design a feedforward controller for a twin-screw food extrusion process to reduce the effect of feed rate and feed moisture content variations on the die pressure. [16] uses second-order transfer functions while emphasizing the difficulty in implementing those types

of model-based controllers due to the strong influence of all the manipulated inputs and measurable process variables. Therefore, very intelligent controllers need to be constructed for extrusion cooking process based on a control algorithm developed from process experience. We mention that the transport delays, the strong interactions and the non-linearities make it difficult to control such systems with *PID* controllers. Predictive controllers might offer better performances but are somewhat difficult to implement [20].

In the present work, we consider the stabilization of the die pressure to the desired setpoint in a food extrusion process. The controller is constructed based on a bi-zone model derived from the computation of the conservation of mass in the extruder under the assumption of constant temperature and viscosity. A Geometric decomposition of the extruder length in *Partially Filled Zone (PFZ)* and *Fully Filled Zone (FFZ)* allows to describe the process by a transport equation and a pressure-gradient equation defined on complementary time varying spatial domains. The domains are coupled by a moving interface whose dynamics is governed by an *ODE* representing the total mass balance in an extruder. We propose suitable feedback control laws together with practical measurements as output such that the solution of the closed-loop system converges to a desired steady-state or equilibrium asymptotically.

The stabilization problems for hyperbolic systems has been widely studied in the literature. The first approach relies on careful analysis of the classical solutions along the characteristics. We refer to Greenberg and Li [10] in the case of second-order system of conservation laws and more general situations on *n*th-order systems by Li in [15].

Another approach based on Lyapunov techniques was introduced by Coron et al. in [3]. This approach was improved in [2] where a strict Lyapunov function in terms of Riemann invariants was constructed and its time derivative can be made negative definite by choosing properly the boundary conditions. The Lyapunov function is very useful to analyze nonlinear hyperbolic systems of conservation laws because of its robustness, see [1], [2], [4], [5], [21] for a wide range of applications to various models. Among which, we are interested in a physical model for the extrusion process which occurs very often in polymer material and food production.

The main contribution of this paper is to establish the exponential stabilization for the extrusion model under natural feedbacks by a Lyapunov function approach motivated by [2], [4], [21]. The difficulties come from four aspects: 1) The domains on which the conservation laws are defined depend on the solution through the dynamical interface; 2) The nonlinear coupling of the dynamics of the interface and the filling ratio

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in the *PFZ* is not standard; 3) The measurements and the feedbacks are natural, however, some of the measurements are indirect, i.e., the measurements are not a part of solution but given through an indirect relation of solution; 4) The feedback controllers are linear and proportional to the measurements, but highly nonlinear with respect to the state variables.

The organization of the paper is as follows. In Section II, we give the physical description of the model. The main result on stabilization (Theorem 1) and its proof are given in Section III. Numerical simulations are provided with a comparison between the proposed approach and linear PI feedback controller based on the unmeasurable state variable in Section IV. Finally in Section V, we give our conclusion and some perspectives in control of the extrusion process.

II. DESCRIPTION OF THE MODEL

An extruder is a process used for manufacturing objects with fixed shapes and specific properties, see Fig. 1. One or two Archimedean screws are rotating inside the barrel in order to convect the extruded material from the feed to the die exit. In food or polymers extrusion processes, the ultimate control systems involved manipulation of screw speed, feed rate, and barrel temperature for the regulation of moisture content, temperature and viscosity of the finite product, residence time and die flow. In this paper, we consider the mass

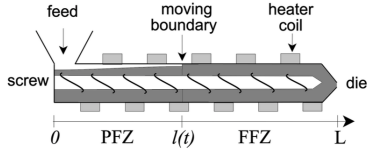


Fig. 1. A single-screw screw extruder

balances model [7], [8] motivated by [13], [14] for cooking extrusion process. In this case, the material convection along the extruder chamber of length L is described in two zones: the *PFZ* ($[0, l(t)]$ in space) and a *FFZ* ($[l(t), L]$ in space) separated by a moving interface $l(t)$. The *PFZ* and the *FFZ* appear due to the die resistance that provokes an accumulation phenomena and high pressure need to be built-up to evict the extrudate out of the die. By the mass conservation principle the convection in the *PFZ* is described by the evolution of the filling ratio $f_p(t, x)$ for a homogeneous melt. The melt convection speed in the *PFZ*, namely, α_p depends on the screw speed $N(t)$ whereas the *FFZ* transport velocity is related to the die pressure $P(t, L)$: $\alpha_p \neq \alpha_f$. Under the assumption of constant viscosity η along the extruder, the dynamics of the moving interface $l(t)$ is governed by an *ODE* arising from the difference of the convection speed in the two regions. The flow rate in the *FFZ* is constant and equal to the die flow rate $F_d(t)$ which is proportional to the pressure difference $\Delta P(t) := P(t, L) - P_0$ where P_0 denotes the atmospheric pressure. For more detailed physical description of the model and definition of all the parameters, one can refer to [7], [8].

In this work, the stabilization of $(l(t), f_p(t, x))$ with the help of the actuated screw speed $N(t)$ and inlet flow rate $F_{in}(t)$

is established based on feedbacks that depend on the pressure difference $\Delta P(t)$ that is a practically useful measurement for the system. Considering the following change of variables

$$x \mapsto y = \frac{x}{l(t)} \text{ in } PFZ \text{ and } x \mapsto y = \frac{x - l(t)}{L - l(t)} \text{ in } FFZ, \quad (1)$$

respectively, the time varying domains ($[0, l(t)]$, $[l(t), L]$) can be transformed to the fixed domain $[0, 1]$ in space. For the sake of simplicity, we still denote by x the space variable instead of y . More precisely, we consider the stabilization problem for the corresponding normalized system on the spatial domain $[0, 1]$. The interface dynamics which arises from a total mass balance writes

$$\begin{cases} \dot{l}(t) = \mathcal{F}(l(t), N(t), f_p(t, 1)), & \text{in } \mathbb{R}^+ = (0, \infty), \\ l(0) = l^0, \end{cases} \quad (2)$$

where

$$\mathcal{F}(l(t), N(t), f_p(t, 1)) = \frac{K_d \Delta P(t) - \rho_o V_{eff} N(t) f_p(t, 1)}{\rho_o S_{eff} (1 - f_p(t, 1))}, \quad (3)$$

K_d , ρ_o , η denote the die conductance, the melt density and the viscosity, respectively. V_{eff} and S_{eff} are the effective volume and section between a screw element and the extruder barrel, respectively. Assuming a constant viscosity along the extruder (the isothermal case), the relation

$$\Delta P(t) = \mathcal{P}(l(t), N(t)) := \frac{\eta \rho_o V_{eff} N(t) (L - l(t))}{B \rho_o + K_d (L - l(t))} \quad (4)$$

is determined by integrating the pressure-gradient equation corresponding to the momentum balance in the *FFZ* and considering a pressure continuity coupling relation at the normalized interface, namely, $P(0, t) = P_0$ in the *PFZ* [8]. The filling ratio in the *PFZ* writes

$$\begin{cases} \partial_t f_p(t, x) + \alpha_p \partial_x f_p(t, x) = 0, & \text{in } \mathbb{R}^+ \times (0, 1), \\ f_p(0, x) = f_p^0(x), & \text{in } (0, 1), \\ f_p(t, 0) = \frac{F_{in}(t)}{\rho_o V_{eff} N(t)}, & \text{in } \mathbb{R}^+, \end{cases} \quad (5)$$

where

$$\alpha_p = \frac{\zeta N(t) - x \mathcal{F}(l(t), N(t), f_p(t, 1))}{l(t)}. \quad (6)$$

III. MAIN RESULT AND ITS PROOF

Let us define the constant equilibrium (l_e, N_e, f_{p_e}) by $\mathcal{F}(l_e, N_e, f_{p_e}) = 0$. Thanks to (3) and (4), for any fixed constants $l_e \in (0, L)$ and $N_e > 0$, it is equivalent to assign

$$f_{p_e} = \frac{K_d (L - l_e)}{B \rho_o + K_d (L - l_e)} \in (0, 1). \quad (7)$$

Correspondingly, ΔP_e , α_{p_e} and F_{in_e} are given by $\Delta P_e = \mathcal{P}(l_e, N_e)$, $\alpha_{p_e} = \frac{\zeta N_e}{l_e}$, $F_{in_e} = \rho_o V_{eff} N_e f_{p_e}$. Denote the difference $\bar{l}(t) := l(t) - l_e$, $\bar{N}(t) := N(t) - N_e$, $\bar{f}_p(t, x) := f_p(t, x) - f_{p_e}$, $\bar{F}_{in}(t) := F_{in}(t) - F_{in_e}$, $\bar{\Delta P}(t) := \Delta P(t) - \Delta P_e$ and the constants

$$\begin{cases} (a_1, a_2, a_3) = \left(\frac{\partial \mathcal{F}}{\partial l}, \frac{\partial \mathcal{F}}{\partial N}, \frac{\partial \mathcal{F}}{\partial f_p} \right) \Big|_{(l_e, N_e, f_{p_e})}, \\ (b_1, b_2) = \left(\frac{\partial \mathcal{P}}{\partial l}, \frac{\partial \mathcal{P}}{\partial N} \right) \Big|_{(l_e, N_e)}. \end{cases} \quad (8)$$

Then, it follows from (3),(4) and (7)-(8) that

$$a_1 = \frac{-K_d V_{eff} B \rho_0 N_e}{S_{eff}(1 - f_{pe})[B \rho_0 + K_d(L - l_e)]^2} < 0, \quad a_2 = 0. \quad (9)$$

The feedback law that we use is the following one:

$$\bar{N}(t) = k_1 \cdot \Delta \bar{P}(t), \quad \bar{F}_{in}(t) = k_2 \cdot \Delta \bar{P}(t), \quad (10)$$

where $\Delta P(t)$, thus $\Delta \bar{P}(t)$, is measurable. The aim of stabilization is to find constants $k_1, k_2 \in \mathbb{R}$ such that the closed-loop system (2) and (5) with feedback (10) is asymptotically stable, i.e., $(\bar{l}(t), \bar{f}_p(t, \cdot)) \rightarrow 0$ as $t \rightarrow \infty$.

Concerning the well-posedness of the Cauchy problem (2) and (5) with feedback (10), we have the following proposition.

Proposition 1. *Let $k_1, k_2 \in \mathbb{R}$ be fixed. There exists $\varepsilon > 0$ such that for any $l^0 \in \mathbb{R}$, $f_p^0 \in H^2(0, 1)$ satisfying $|l^0 - l_e|^2 + \|f_p^0(\cdot) - f_{pe}\|_{H^2(0,1)}^2 \leq \varepsilon$, and the compatibility conditions at the point $(t, x) = (0, 0)$, system (2) and (5) with (10) has a unique solution $(l, f_p) \in W^{1,\infty}([0, T]) \times C^0([0, T]; H^2(0, 1))$ for some $T \in (0, \infty]$. Moreover, if $|l(t) - l_e|^2 + \|f_p(t, \cdot) - f_{pe}\|_{H^2(0,1)}^2 \leq \varepsilon$ for all $t \in [0, T]$, then $T = \infty$.*

Remark 1. *The compatibility conditions at the point $(t, x) = (0, 0)$ are the following:*

$$f_p^0(0) = \frac{F_{in}(0)}{\rho_o V_{eff} N(0)}; \quad (11)$$

$$\frac{\dot{F}_{in}(0)N(0) - F_{in}(0)\dot{N}(0)}{\rho_o V_{eff} N(0)} + \frac{\zeta N(0)}{l^0} f_p^{0'}(0) = 0, \quad (12)$$

where $N(0), F_{in}(0)$ are determined by (4), (10) with $l(0) = l^0$, while $\dot{N}(0), \dot{F}_{in}(0)$ are determined by differentiating (4) and (10) together with $\dot{l}(0) = F(l^0, N(0), f_p^0(1))$.

The proof of Proposition 1 is based on fixed point argument and one can refer to [8] for the well-posedness of the corresponding open-loop system. Our main result on stabilization of the interface position $l(t)$ and the filling ratio $f_p(t, x)$ is the following theorem.

Theorem 1. *Let $k_1, k_2 \in \mathbb{R}$ be such that*

$$\left| \frac{a_3 b_1 (k_2 - f_{pe} \rho_o V_{eff} k_1)}{\rho_o V_{eff} N_e (1 - k_1 b_2)} \right| < |a_1|, \quad (13)$$

where a_1, a_3, b_1, b_2 are given in (8). Then, the nonlinear system (2) and (5) is locally exponentially stable under the feedback (10), i.e., there exist constants $\varepsilon > 0$, $M > 0$ and $\omega > 0$ such that for any $l^0 \in \mathbb{R}$, $f_p^0 \in H^2(0, 1)$ satisfying

$$|l^0 - l_e|^2 + \|f_p^0(\cdot) - f_{pe}\|_{H^2(0,1)}^2 \leq \varepsilon, \quad (14)$$

and the compatibility conditions at the point $(t, x) = (0, 0)$, the solution of (2) and (5) with (10) satisfies

$$\begin{aligned} & |l(t) - l_e|^2 + \|f_p(t, \cdot) - f_{pe}\|_{H^2(0,1)}^2 \\ & \leq M e^{-\omega t} \left(|l^0 - l_e|^2 + \|f_p^0(\cdot) - f_{pe}\|_{H^2(0,1)}^2 \right), \quad \forall t \geq 0. \end{aligned} \quad (15)$$

Remark 2. *The measurement on $\Delta P(t)$ is of practical reason, thus the feedback (10) is indirect in the sense that the measurements are made not on the state $(l(t), f_p(t, x))$ itself.*

The feedback controllers are linear and proportional to the measurements, but highly nonlinear with respect to the state.

Remark 3. *In practice, the controller saturation problems should be taken in account. Therefore, the gains k_1, k_2 satisfying (13) should be chosen properly under some natural constraints, for instance, large k_1, k_2 might result into too large screw speed and too large inlet flow. Additionally, large k_1, k_2 might also reduce the range of the stabilizable state (typically, the value $\varepsilon > 0$). On the other hand, small k_1, k_2 might result into too slow controls so that the time to reach the set point would be too large.*

Proof of Theorem 1: By definition of the equilibrium (l_e, N_e, f_{pe}) and the constants $(a_1, a_2, a_3, b_1, b_2)$, it is easy to get by expansion and $a_2 = 0$ in (9) that

$$\begin{aligned} \mathcal{F}(l(t), N(t), f_p(t, 1)) &= (a_1 + o(1)) \bar{l}(t) + o(1) \bar{N}(t) \\ &\quad + (a_3 + o(1)) \bar{f}_p(t, 1), \end{aligned} \quad (16)$$

$$\Delta \bar{P}(t) = (b_1 + o(1)) \bar{l}(t) + (b_2 + o(1)) \bar{N}(t). \quad (17)$$

Furthermore, it follows from (10) and (17) that

$$\Delta \bar{P} = \left(\frac{b_1}{1 - k_1 b_2} + o(1) \right) \bar{l}(t), \quad (18)$$

$$\bar{F}_{in}(t) = \left(\frac{k_2 b_1}{1 - k_1 b_2} + o(1) \right) \bar{l}(t), \quad (19)$$

$$\bar{N}(t) = \left(\frac{k_1 b_1}{1 - k_1 b_2} + o(1) \right) \bar{l}(t), \quad (20)$$

where $o(1)$ represents various terms which tend to 0 when $|(\bar{l}(t), \bar{N}(t), \bar{f}_p(t, 1))| \rightarrow 0$.

Then we construct the Lyapunov functions relying on the Lemmas 1, 2, 3 whose proofs are given in Appendix.

Let

$$V_0(t) = \bar{l}^2(t) \quad (21)$$

$$V_1(t) = \int_0^1 e^{-\gamma_1 x} \bar{f}_p^2(t, x) dx, \quad (22)$$

$$V_2(t) = \int_0^1 e^{-\gamma_2 x} f_{px}^2(t, x) dx, \quad (23)$$

$$V_3(t) = \int_0^1 e^{-\gamma_3 x} f_{p_{xx}}^2(t, x) dx, \quad (24)$$

where $\gamma_i > 0$ ($i = 1, 2, 3$) are constants to be chosen later.

Lemma 1. *There exist positive constants $A_1, \gamma_1, \beta_0, \beta_1, \delta_1$ such that the following estimates hold for every solution to system (2) and (5) with (10)*

$$\begin{aligned} \dot{V}_0(t) + A_1 \dot{V}_1(t) &\leq -(\beta_0 + o(1)) V_0(t) - (\beta_1 + o(1)) V_1(t) \\ &\quad - (\delta_1 + o(1)) \bar{f}_p^2(t, 1). \end{aligned} \quad (25)$$

Differentiating (5) with respect to x , we get

$$\begin{cases} f_{p_{xt}} + \alpha_p f_{p_{xx}} + \alpha_{px} f_{px} = 0, \\ f_{px}(0, x) = f_p^{0'}(x), \\ f_{px}(t, 0) = \frac{-f_{pt}(t, 0)}{\alpha_p|_{x=0}}, \end{cases} \quad (26)$$

where

$$f_{pt}(t, 0) = \frac{\dot{F}_{in}(t)N(t) - F_{in}(t)\dot{N}(t)}{\rho_o V_{eff} N^2(t)}. \quad (27)$$

Lemma 2. *There exist positive constants $\gamma_2, \beta_2, \delta_2, \theta_2$ such that the following estimate holds for every solution to system (2) and (5) with (10)*

$$\begin{aligned} \dot{V}_2(t) \leq & -(\beta_2 + o(1))V_2(t) - (\delta_2 + o(1))f_{p_x}^2(t, 1) \\ & + \theta_2(V_0(t) + \bar{f}_p^2(t, 1)). \end{aligned} \quad (28)$$

By differentiating (26), note that $\alpha_{p_{xx}} = 0$, we derive that

$$\begin{cases} f_{p_{xxt}} + \alpha_p f_{p_{xxx}} + 2\alpha_{p_x} f_{p_{xx}} = 0, \\ f_{p_{xx}}(0, x) = f_p^{0''}(x), \\ f_{p_{xx}}(t, 0) = \frac{-f_{p_{xt}}(t, 0) + \alpha_{p_x}|_{x=0} f_{p_x}(t, 0)}{\alpha_p|_{x=0}}, \end{cases} \quad (29)$$

where

$$f_{p_{xt}}(t, 0) = \frac{d}{dt} \left(\frac{-1}{\alpha_p|_{x=0}} \cdot \frac{\dot{F}_{in}(t)N(t) - F_{in}(t)\dot{N}(t)}{\rho_o V_{eff} N^2(t)} \right). \quad (30)$$

Lemma 3. *There exist positive constants $\gamma_3, \beta_3, \delta_3, \theta_3$ such that the following estimate holds for every solution to system (2) and (5) with (10)*

$$\begin{aligned} \dot{V}_3(t) \leq & -(\beta_3 + o(1))V_3(t) - (\delta_3 + o(1))f_{p_{xx}}^2(t, 1) \\ & + \theta_3(V_0(t) + \bar{f}_p^2(t, 1) + f_{p_x}^2(t, 1)). \end{aligned} \quad (31)$$

Finally, let the Lyapunov function be

$$L(t) = V_3(t) + A_3(V_2(t) + A_2(V_0(t) + A_1 V_1(t))), \quad (32)$$

where $A_1 > 0$ is such that (25) holds and $A_2, A_3 > 0$ will be chosen later. Obviously, $L(t)$ is equivalent to $\bar{l}^2(t) + \|\bar{f}_p(t, \cdot)\|_{H^2(0,1)}^2$. Then, by (25), (28), (31) and (32), one can choose $A_2 > 0$ and $A_3 > 0$ successively large such that

$$\dot{L}(t) \leq -(\beta + o(1))L(t) \quad (33)$$

for some constant $\beta > 0$. We assume in a priori that

$$|(\bar{l}(t), \bar{N}(t), \bar{f}_p(t, 1))| \leq \varepsilon_0 \quad (34)$$

for some small $\varepsilon_0 > 0$ such that $|o(1)| \leq \frac{\beta}{2}$ in (33). Then $\dot{L}(t) \leq -\frac{\beta}{2}L(t)$, thus $L(t) \leq L(0)e^{-\frac{\beta}{2}t}$. Thanks to the assumption (14), (34) can be satisfied for all $t \geq 0$ if $\varepsilon > 0$ is small enough. The proof of Theorem 1 is thus complete. \square

Remark 4. *The weight as $e^{-\gamma_i x}$ is essential to get a strict Lyapunov function. One can refer to the stabilization results by such weighted Lyapunov functions, for quite general linear hyperbolic systems in [21]; for one dimensional Euler equation with variable coefficients in [11]; for a conservation law with nonlocal velocity in [4].*

IV. SIMULATIONS

Computing the time integration of the semi-discretized transport equations by finite volume with ODE45 routine of MATLAB, the stability result is achieved under the assumptions of Theorem 1.

• Initial conditions:

$$f_p^0(x) = 0.6905 + 0.025(1 - \cos(\pi x)) + 0.0117 \sin(\pi x)$$

$$l^0 = 1.5 \text{ m or } l^0 = 0.6 \text{ m}$$

• Setpoint values: $l_e = 1.37 \text{ m}$, $N_e = 220$, $f_{pe} = 0.6$

• Gain values: $k_1 = 0.01$, $k_2 = 0.0001$

For $l_e \in (0, L)$ and $N_e > 0$, f_{pe} is uniquely determined by (7). The gain k_1, k_2 is chosen to satisfy (13). The initial data ($l^0, f_p^0(x)$) are chosen to satisfy the compatibility conditions (11) and (12) as well as the feedback law (10). While in simulations for the case $l^0 = 0.6 \text{ m}$, (12) is satisfied with an error of order 10^{-20} .

The simulations show the comparison between the controllers (10) and the linear PI feedback of the unmeasurable state $l(t)$ defined as

$$\bar{N}(t) = \frac{k_1 b_1}{1 - k_1 b_2} \bar{l}(t) + k_1 \int \bar{l}(s) ds, \quad (35)$$

$$\bar{F}_{in}(t) = \frac{k_2 b_1}{1 - k_1 b_2} \bar{l}(t) + k_2 \int \bar{l}(s) ds. \quad (36)$$

The proportional PI controllers (35)-(36) are motivated by the estimated linear control laws (19)-(20). We emphasize that the controller (10) is based on the accessible output pressure measurement but remain highly nonlinear with respect to the state variables.

Figures 2, 3 and 4 show that the trends of the state $l(t)$, the output pressure, $\Delta P(t)$, and the control actions $F_{in}(t)$ and $N(t)$ are similar when the initial condition are closed to the equilibrium $l_e = 1.37 \text{ m}$, namely for $l^0 = 1.5 \text{ m}$. For large initial data, namely, $l^0 = 0.6 \text{ m}$, the pressure feedback control law (10) allows faster convergence and hence stands as a better approach due to the inaccessibility of the state $l(t)$ and the resulting performances. Moreover, Figure 4 shows that for large initial data the compatibility conditions are not satisfied for the PI controller. One should mention that a better tuning of the PI gains might results in better performances.

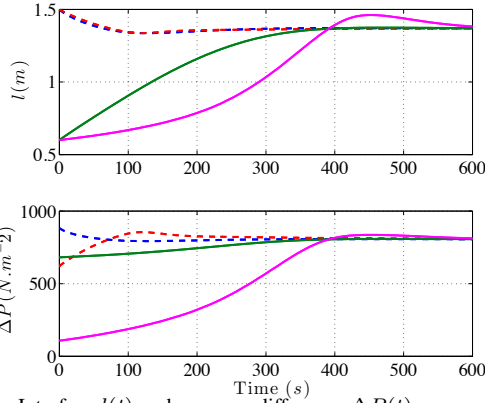
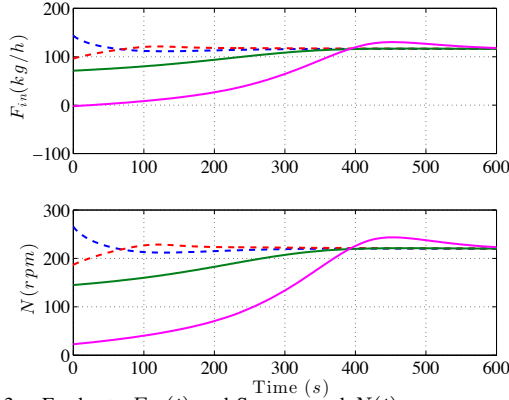
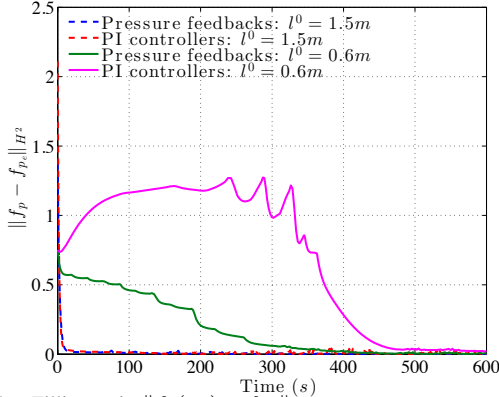
V. CONCLUSION

In this paper, we study the stabilization of a physical model for the extrusion process, which is described by conservation laws coupled through a dynamical interface. The exponential stabilization is obtained for the closed-loop system with natural but nonlinear feedback controls through indirect measurements. The proof relies on Lyapunov approach. Numerical simulations are made as supplementary to the results with a comparison between the proposed approach and linear PI feedback controller based on the unmeasurable state variable. As a future work, it would be interesting to study the controllability of boundary profile, i.e., to reach the desired moisture and temperature at the die under suitable controls. This problem is rather challenging for mathematical theory but also very useful in applications. Moreover, the proposed result might be extended to a PI controller for complex screw extrusion systems that include unmeasured disturbances.

APPENDIX

Physical definition of the parameters

$L = 2 \text{ m}$	Extruder Length
$B = 2.4 \times 10^{-5} \text{ m}^4$	Geometric parameter
$K_d = 2 \times 10^{-2} \text{ m}^3$	Geometric parameter
$\zeta = 0.003 \text{ m}$	Screw Pitch
$\eta = 125 \text{ Pa s}^{-1}$	Melt viscosity
$\rho_o = 350 \text{ kg m}^{-3}$	Melt density
$S_{eff} = 0.014 \text{ m}^2$	Effective area
$V_{eff} = \zeta S_{eff}$	Effective volume


 Fig. 2. Interface $l(t)$ and pressure difference $\Delta P(t)$

 Fig. 3. Feed rate $F_{in}(t)$ and Screw speed $N(t)$

 Fig. 4. Filling ratio $\|f_p(t, \cdot) - f_{p_e}\|_{H^2(0,1)}$

Proof of Lemma 1: Differentiating $V_0(t)$ with respect to t and using (2), (16) and (20), one easily gets that

$$\begin{aligned} \dot{V}_0(t) &= 2\bar{l}(t) \cdot \mathcal{F}(l(t), N(t), f_p(t, 1)) \\ &= (2a_1 + o(1))\bar{l}^2(t) + (2a_3 + o(1))\bar{l}(t)\bar{f}_p(t, 1). \end{aligned} \quad (37)$$

On the other hand, (6) and (16) yield that

$$\alpha_p = \alpha_{p_e} + o(1), \quad \alpha_{p_x} = o(1). \quad (38)$$

Differentiating $V_1(t)$ gives, from (22), (38) and integration by parts, that

$$\begin{aligned} \dot{V}_1(t) &= BT_1 + \int_0^1 (-\gamma_1 \alpha_p + \alpha_{p_x}) e^{-\gamma_1 x} \bar{f}_p^2(t, x) dx \\ &= BT_1 - (\gamma_1 \alpha_{p_e} + o(1))V_1(t), \end{aligned} \quad (39)$$

where

$$BT_1 = (-e^{-\gamma_1} \alpha_{p_e} + o(1))\bar{f}_p^2(t, 1) + (\alpha_{p_e} + o(1))\bar{f}_p^2(t, 0). \quad (40)$$

Note that by (5),(20)-(19), we have

$$\bar{f}_p(t, 0) = \frac{F_{in}(t)}{\rho_o V_{eff} N(t)} - f_{p_e} = (\theta_1 + o(1))\bar{l}(t), \quad (41)$$

where $\theta_1 = \frac{b_1(k_2 - f_{p_e} \rho_o V_{eff} k_1)}{\rho_o V_{eff} N_e (1 - k_1 b_2)}$. Combining (38), (39), (40), (41), we get consequently

$$\begin{aligned} \dot{V}_1(t) &= -(\gamma_1 \alpha_{p_e} + o(1))V_1(t) - (e^{-\gamma_1} \alpha_{p_e} + o(1))\bar{f}_p^2(t, 1) \\ &\quad + (\alpha_{p_e} \theta_1^2 + o(1))V_0(t). \end{aligned} \quad (42)$$

By (9) and the assumption (13), it is easy to get the existence of $A_1 > 0$ and $\gamma_1 > 0$ (suitably small) such that

$$\begin{pmatrix} 2a_1 + A_1 \alpha_{p_e} \theta_1^2 & a_3 \\ a_3 & -A_1 e^{-\gamma_1} \alpha_{p_e} \end{pmatrix} \quad (43)$$

is negative definite. This concludes the proof of Lemma 1. \square

Proof of Lemma 2. Differentiating (4) and (10) with respect to t gives that,

$$\begin{cases} \dot{N}(t) = k_1 \cdot \Delta \dot{P}(t), & \dot{F}_{in}(t) = k_2 \cdot \Delta \dot{P}(t), \\ \Delta \dot{P}(t) = \frac{\partial \mathcal{P}}{\partial l} \Big|_{(l(t), N(t))} \dot{l}(t) + \frac{\partial \mathcal{P}}{\partial N} \Big|_{(l(t), N(t))} \dot{N}(t). \end{cases} \quad (44)$$

Then it follows from (2), (8), (16), (18) and (44) that

$$\begin{aligned} \Delta \dot{P}(t) &= \frac{\frac{\partial \mathcal{P}}{\partial l} \Big|_{(l(t), N(t))}}{1 - k_1 \frac{\partial \mathcal{P}}{\partial N} \Big|_{(l(t), N(t))}} \cdot \dot{l}(t) \\ &= O(1)(\bar{l}(t) + \bar{N}(t) + \bar{f}_p(t, 1)), \end{aligned} \quad (45)$$

where $O(1)$ denotes various terms which are uniformly bounded when $|(\bar{l}(t), \bar{N}(t), \bar{f}_p(t, 1))| \rightarrow 0$.

Combining (20), (26)-(27) and (44)-(45), we get easily that

$$f_{p_x}(t, 0) = O(1)(\bar{l}(t) + \bar{f}_p(t, 1)). \quad (46)$$

Differentiating (23) results in, by (26) and (38), that

$$\begin{aligned} \dot{V}_2(t) &= BT_2 + \int_0^1 (-\gamma_2 \alpha_p - \alpha_{p_x}) e^{-\gamma_2 x} f_{p_x}^2(t, x) dx \\ &= BT_2 + (-\gamma_2 \alpha_{p_e} + o(1))V_2(t), \end{aligned} \quad (47)$$

where

$$\begin{aligned} BT_2 &= (-e^{-\gamma_2} \alpha_{p_e} + o(1))f_{p_x}^2(t, 1) \\ &\quad + (\alpha_{p_e} + o(1))f_{p_x}^2(t, 0). \end{aligned} \quad (48)$$

Thanks to (38) and (46), (48) can be rewritten as

$$\begin{aligned} BT_2 &= (-e^{-\gamma_2} \alpha_{p_e} + o(1))f_{p_x}^2(t, 1) \\ &\quad + O(1)(\bar{l}^2(t) + \bar{f}_p^2(t, 1)), \end{aligned} \quad (49)$$

which ends the proof of Lemma 2 with (47). \square

Proof of Lemma 3. Differentiating (24) gives, by (29) and (38), that

$$\begin{aligned} \dot{V}_3(t) &= BT_3 + \int_0^1 [-\gamma_3 \alpha_p - 3\alpha_{p_x}] e^{-\gamma_3 x} f_{p_{xx}}^2 dx \\ &= BT_3 + (-\gamma_3 \alpha_{p_e} + o(1))V_3(t), \end{aligned} \quad (50)$$

where

$$BT_3 = - (e^{-\gamma_3} \alpha_{p_e} + o(1)) f_{p_{xx}}^2(t, 1) + (\alpha_{p_e} + o(1)) f_{p_{xx}}^2(t, 0). \quad (51)$$

In order to estimate $f_{p_{xx}}(t, 0)$ or $f_{p_{xt}}(t, 0)$, essentially we need only to estimate $\bar{F}_{in}(t)$ and $\bar{N}(t)$, according to (30) and (44). On the other hand, (44) simply yields that

$$\bar{N}(t) = k_1 \Delta \bar{P}(t), \quad \bar{F}_{in}(t) = k_2 \Delta \bar{P}(t). \quad (52)$$

Therefore, from (2), (16), (44) and (45), we have

$$\Delta \bar{P}(t) = O(1) \left(\bar{l}(t) + \bar{N}(t) + \bar{f}_p(t, 1) + f_{p_t}(t, 1) \right). \quad (53)$$

From (30) to (53), we get

$$f_{p_{xt}}(t, 0) = O(1) \left(\bar{l}(t) + \bar{N}(t) + \bar{f}_p(t, 1) + f_{p_t}(t, 1) \right). \quad (54)$$

Combining (5), (20), (29) and (54), we get further

$$f_{p_{xx}}(t, 0) = O(1) \left(\bar{l}(t) + \bar{f}_p(t, 1) + f_{p_x}(t, 1) \right). \quad (55)$$

By (38) and (55), (51) becomes

$$BT_3 = - (e^{-\gamma_3} \alpha_{p_e} + o(1)) f_{p_{xx}}^2(t, 1) + O(1) (\bar{l}^2(t) + \bar{f}_p^2(t, 1) + f_{p_x}^2(t, 1)). \quad (56)$$

This implies the conclusion of Lemma 3. \square

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