Boundary Control of Nonlinear ODE/Wave PDE Systems with Spatially-Varying Propagation Speed

Xiushan Cai and Mamadou Diagne

***Abstract*—We consider the boundary control of a nonlinear ODE actuated through a wave equation whose propagation speed is spatially-varying. The ODE state is driven by the uncontrolled boundary of the wave equation. We design a nonlinear backstepping compensator to enable global asymptotic stability of the closed-loop system. We deduce the controller design and the stability proof by introducing a two-step backstepping transformation. The first transformation recasts the original system into a coupled 2×2 first-order hyperbolic system with spatially-varying coefficients cascading into a nonlinear ODE. The second transformation is used in the design of a compensator for the resulting cascaded system. Our design offers a global stability result that is guaranteed assuming that the spatially-varying propagation speed is continuously differentiable and positive. Moreover, for nonlinear systems, our result is the first contribution enabling actual compensation of actuator delays governed by a coupled first-order hyperbolic PDEs induced by a wave PDE dynamics with spatially-varying propagation speed. The validity of the proposed controller is illustrated by the benchmark system controlled via a cable.**

***Index Terms*—Nonlinear system, wave PDE, predictor feedback control, spatially-varying coefficients.**

I. Introduction

Recent studies using backstepping control technique have enabled the stabilization of nonlinear ordinary differential equation (ODE) systems with input delays that depend on the ODE state in [1], [2], [3], [4], as well as the uncontrolled-or controlled-boundary value of the partial differential equation (PDE) state in [5], [6], [7], [8]. Later, the method has been extended to deal with the stabilization problem of nonlinear systems with actuator dynamics governed by wave PDE with moving boundary that depends on the ODE state [9], [10], [11]. The method has also been employed to control transport PDE-ODE cascades with delayed input [12], as well as with state-dependent propagation speed [13], [14].

The result of [15] on the boundary feedback control of PDE-ODE cascaded systems highlighted the potential of the PDE control for various physical systems. Along the same lines, in [16], the stabilization of a linear ODE whose actuator dynamics is governed by a first-order linear hyperbolic PDE is achieved via backstepping design. Using a two-step backstepping method, boundary control of linear ODE with linear 2×2 hyperbolic systems with spatially-varying coefficients and dynamic boundary conditions has been established in [18] and results dealing with the inverse optimal control for systems with input delays can be found in [19], [20], [21], [22]. The cascaded system consisting of an ODE actuated via a hyperbolic PDE is relevant to many engineering problems including metal rolling processes [23], and metal cutting processes [24], vehicular traffic flow [25], moisture in convective flows [26], transport phenomena in gasoline engines [27], [28], commercial fuels by blending [29].

The particular case of actuator dynamics governed by a wave PDE is quite interesting and has been proven to enable stabilization of stick-slip instabilities and bit-bouncing phenomena in oil drilling processes [13], [30], [31], [32], [33]. These phenomena lead to growing torsional and axial vibrations resulting from the complex interaction between the drill bit and the deeply cracked rock when operating drill strings [34]. In fact, neglecting the damping coefficients between the mud and the pipe, the axial and torsional excitations of the drill string can be described by a wave PDE cascading into a nonlinear ODE governing the dynamic boundary condition at the bit-rock point of contact. Through linear and bilinear matrix inequalities techniques, feedback controllers are established guaranteeing ultimate boundedness of the system trajectories and leading consequently to the suppression of harmful dynamics in drilling system [35]. The dynamics of a flexible cable crane with a load can be expressed by a wave PDE/ nonlinear ODE cascaded system [36]. One should mention recent advances achieved in stabilizing 2×2 coupled hyperbolic PDEs in cascade with linear ODEs to cancel oscillations of tension and cage in dual-cable mining elevators [37] and deep-sea construction [38].

In the present work, we deal with a problem that is similar but not equivalent to the model used to stabilize oscillations in drilling systems. The main goal of this contribution is to design a compensator for the delay induced by the wave PDE with a spatially distributed propagation speed. Though many similarities might be found in the existing literature regarding the stability proof rational, the derived predictor state cannot be obtained by extending any of the existing results based on PDE backstepping design. The key difference between the structure of the considered system and the one investigated in [9] arises after transforming the original wave PDE into a linear 2×2 hyperbolic system. Clearly, in [9], the resulting transport PDEs are decoupled while the presence of a spatially distributed coefficient induces strongly coupled hyperbolic

X. Cai is with the College of Mechatronics and Control Engineering, Hubei Normal University, Huangshi, 435002, China, and the College of Physics and Electronic Information Engineering, Zhejiang Normal University, Jinhua, 321004, China. e-mail: xiushancai@163.com.

M. Diagne is with the Department of Mechanical Aerospace and Nuclear Engineering, Rensselaer Polytechnic Institute, Troy, NY 12180, USA. e-mail: diagnm@rpi.edu

This work is supported in part by the National Natural Science Foundation of China (61773350, 62072164), and the Natural Science Foundation of Zhejiang Province of China (LY17F030001).

PDEs with spatially distributed coefficients cascading with a nonlinear ODE in our case. The strong coupling makes the design of an actuator dynamics compensator non-trivial and different from [9] and [10], [11], which deal with constant propagation speed. Boundary controllers for consisting of a linear ODE in cascade with coupled hyperbolic PDEs can be found in [37], [38]. Mathematically, *a novel two-step backstepping transformation* is employed to derive a non-standard target system whose stability is established using a Lyapunov argument. The resulting boundary controller is a predictor-feedback control law, which compensates the wave actuator dynamics and guarantees global asymptotic stability of the closed-loop system.

This paper is organized as follows: the system's description, the main result are presented in Section II. The transformation of the original system into a 2×2 coupled linear hyperbolic PDEs is in Section III. A first backstepping transformation is introduced in Section IV-A. A second backstepping transformation is employed to design a compensator for the resulting decoupled system in Section IV-B. The stability analysis of the target system is established in Section V. Stability of the original system is stated in Section VI. Finally, an example is provided in Section VII, and concluding remarks are emphasized in Section VIII.

Notation. We use the common definitions of class $\mathscr{K}$, $\mathscr{K}_\infty$, $\mathscr{K}\mathscr{L}$ functions from [29]. For an n-vector, $|\cdot|$ denotes the usual Euclidean norm. For a scalar function $u(\cdot,t)$, we denote with $\|u(t)\|_\infty$ its supremum norm, i.e. $\|u(t)\|_\infty = \sup_{x\in[0,L]} |u(x,t)|$.

II. Problem statement and main result

A. System description

We consider a cascaded system consisting of a nonlinear ODE whose actuation path is governed by a wave PDE with spatially-varying propagation speed. The cascaded system has the structure

$$\dot X(t) = f(X(t),u(0,t)), \tag{1}$$
$$\partial_{tt}u(x,t) = v(x)\partial_{xx}u(x,t), \tag{2}$$
$$\partial_x u(0,t) = 0, \tag{3}$$
$$\partial_x u(L,t) = U(t), \tag{4}$$

where $t\geq 0, 0\leq x\leq L$ and $X\in\mathbb{R}^n, u\in\mathbb{R}, U\in\mathbb{R}$ are ODE state, PDE state, and control input, respectively, and $f:\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}^n$ is locally Lipschitz with $f(0,0)=0$, $v:[0,L]\to\mathbb{R}_+$.

We design a predictor control that stabilizes the PDE/ODE cascaded system, under the following Assumptions.

Assumption 1: Propagation speed $v(x)$ is continuously differentiable and positive for all $x\in[0,L]$.

Remark 1: Denote

$$\underline{\upsilon} = \inf_{x\in[0,L]} v(x), \quad \overline{\upsilon} = \sup_{x\in[0,L]} v(x), \tag{5}$$

for all $x\in[0,L]$, the following holds

$$\overline{\upsilon}\geq v(x)\geq \underline{\upsilon} > 0. \tag{6}$$

Assumption 2: System $\dot X = f(X,\kappa(X)+\upsilon)$ is input-to-state stable with respect to υ and the function $\kappa:\mathbb{R}^n\to\mathbb{R}$ is continuously differentiable with locally Lipschitz derivative $\frac{\partial\kappa(X)}{\partial X}$ and it satisfies $\kappa(0)=0$.

Now, consider $\upsilon\in\mathbb{R}$ and define the variables

$$Z(t) = \begin{bmatrix} X(t) \\ u(0,t)\end{bmatrix}, \quad \varphi(Z(t),\upsilon) = \begin{bmatrix} f(X(t),u(0,t)) \\ \upsilon\end{bmatrix}. \tag{7}$$

Assumption 3: System $\dot Z = \varphi(Z,\upsilon)$ is strongly forward/backward complete with respect to υ, that is, there exist smooth positive definite functions R_1, R_2 and class $\mathscr{K}_\infty$ functions $\alpha_1,\cdots,\alpha_6$ such that for all $Z\in R^{n+1}$ and $\upsilon\in R$,

$$\alpha_1(|Z|)\leq R_1(Z) \leq \alpha_2(|Z|) \tag{8}$$
$$\frac{\partial R_1(Z)}{\partial Z}\varphi(Z,\upsilon)\leq R_1(Z)+\alpha_3(|\upsilon|) \tag{9}$$
$$\alpha_4(|Z|)\leq R_2(Z) \leq \alpha_5(|Z|) \tag{10}$$
$$-\frac{\partial R_2(Z)}{\partial Z}\varphi(Z,\upsilon)\leq R_2(Z)+\alpha_6(|\upsilon|). \tag{11}$$

Assumption 4: System $\dot Z = \varphi(Z,\mu(Z)+\upsilon)$ is strongly backward complete with respect to υ, that is, there exist a smooth positive definite function R_3 and class $\mathscr{K}_\infty$ functions α_7, α_8, α_9 such that for $Z\in R^{n+1}$ and $\upsilon\in R$,

$$\alpha_7(|Z|) \leq R_3(Z) \leq \alpha_8(|Z|) \tag{12}$$
$$-\frac{\partial R_3(Z)}{\partial Z}\varphi(Z,\mu(Z)+\upsilon) \leq R_3(Z)+\alpha_9(|\upsilon|). \tag{13}$$

Remark 2: The wave PDE described by (2)–(4) induces an input delay on the nonlinear dynamics of (1). For instance, an actuator dynamics governed by a pure transport PDE with spatially varying coefficient, defined as

$$\dot X(t) = f(X(t),u(0,t)), \tag{14}$$
$$\partial_t u(x,t) = v(x)\partial_x u(x,t), \quad u(L,t)=U(t), \tag{15}$$

is equivalent to the nonlinear system with input delay $\dot X(t) = f(X(t),U(\phi(t)))$, where $\phi(t) = t-\int_0^L v^{-1}(s)ds$.

B. Main results

Suppose the existence of a nominal controller $\kappa:\mathbb{R}^n\to\mathbb{R}$ that stabilizes the delay-free plant, namely, the control law κ is such that the closed-loop system $\dot X(t) = f(X(t),\kappa(X(t)))$ is globally asymptotically stable. The predictor feedback control for system (1)–(4) is given by

$$\begin{aligned} U(t) =& -c_1\frac{e^{\int_0^L \frac{v'(r)}{4v(r)}dr}}{2\sqrt{v(L)}}\left(p_2(L,t)-\kappa(p_1(L,t))\right) \\ &+\frac{e^{\int_0^L \frac{v'(r)}{4v(r)}dr}}{2\sqrt{v(L)}}\frac{\partial\kappa(p_1(L,t))}{\partial p_1}f(p_1(L,t),p_2(L,t)) \\ &-\frac{1}{2\sqrt{v(L)}}\Big(\partial_t u(L,t)-\sqrt{v(L)}\,\partial_x u(L,t)\Big) \\ &+\frac{1}{2\sqrt{v(L)}}\int_0^L K_{11}(L,s)\Big(\partial_t u(s,t)+\sqrt{v(s)}\,\partial_s u(s,t)\Big)ds \\ &+\frac{1}{2\sqrt{v(L)}}\int_0^L K_{12}(L,s)\Big(\partial_t u(s,t)-\sqrt{v(s)}\,\partial_s u(s,t)\Big)ds, \end{aligned} \tag{16}$$

where $p_1 \in \mathbb{R}^n, p_2 \in \mathbb{R}$ are given by

$$p_1(x,t) = X(t) + \int_0^x \frac{f(p_1(y,t),p_2(y,t))}{\sqrt{v(y)}}dy, \tag{17}$$

$$\begin{aligned} p_2(x,t) = u(0,t) + \int_0^x \left(\frac{1}{\sqrt{v(y)}} - \int_y^x \frac{K_{11}(\sigma,y)}{\sqrt{v(\sigma)}}d\sigma \right) \\ \times e^{-\int_0^y \frac{v'(r)}{4v(r)}dr}(\partial_t u(y,t) + \sqrt{v(y)}\,\partial_y u(y,t))dy \\ - \int_0^x \int_y^x \frac{K_{12}(\sigma,y)}{\sqrt{v(\sigma)}}d\sigma \\ \times e^{-\int_0^y \frac{v'(r)}{4v(r)}dr}(\partial_t u(y,t) - \sqrt{v(y)}\,\partial_y u(y,t))dy, \end{aligned} \tag{18}$$

for all $x \in [0,L]$ with initial conditions as

$$p_1(x,0) = X(0) + \int_0^x \frac{f(p_1(y,0),p_2(y,0))}{\sqrt{v(y)}}dy \tag{19}$$

$$\begin{aligned} p_2(x,0) = u(0,0) + \int_0^x \left(\frac{1}{\sqrt{v(y)}} - \int_y^x \frac{K_{11}(\sigma,y)}{\sqrt{v(\sigma)}}d\sigma \right) \\ \times e^{-\int_0^y \frac{v'(r)}{4v(r)}dr}(\partial_t u(y,0) + \sqrt{v(y)}\,\partial_y u(y,0))dy \\ - \int_0^x \int_y^x \frac{K_{12}(\sigma,y)}{\sqrt{v(\sigma)}}d\sigma \\ \times e^{-\int_0^y \frac{v'(r)}{4v(r)}dr}(\partial_t u(y,0) - \sqrt{v(y)}\,\partial_y u(y,0))dy. \end{aligned} \tag{20}$$

The gain c_1 in (16) is an arbitrary constant while the kernel gains K_{11} and K_{12} are solutions to the following gain kernel PDEs:

$$\mathscr{A}(x)\partial_x K(x,s) + \partial_s(K(x,s)\mathscr{A}(s)) = K(x,s)\mathscr{B}(s) \tag{21}$$

$$K(x,x)\mathscr{A}(x) - \mathscr{A}(x)K(x,x) = \mathscr{B}(x) \tag{22}$$

$$K_{11}(x,0) = K_{12}(x,0) \tag{23}$$

$$K_{21}(x,0) = K_{22}(x,0) \tag{24}$$

where (21) is defined on $\{(x,s) : 0 \le s \le x \le 1\}$, and

$$K(x,s) = \begin{bmatrix} K_{11}(x,s) & K_{12}(x,s) \\ K_{21}(x,s) & K_{22}(x,s) \end{bmatrix} \in \mathbb{R}^{2\times 2}, \tag{25}$$

and

$$\mathscr{A}(x) = \begin{bmatrix} \sqrt{v(x)} & 0 \\ 0 & -\sqrt{v(x)} \end{bmatrix}, \mathscr{B}(x) = \begin{bmatrix} 0 & -\frac{v'(x)}{4\sqrt{v(x)}} \\ \frac{v'(x)}{4\sqrt{v(x)}} & 0 \end{bmatrix}. \tag{26}$$

Theorem 1: Consider system (1)–(4) together with the control law (16)–(18), for any initial condition $u(\cdot,0) \in C_1[0,1]$, $u_t(\cdot,0) \in C[0,1]$, which is compatible with the feedback law (16)–(18) and which is such that $u_x(0,0) = 0$. Under Assumptions 1–4, the closed-loop system has a unique solution $X(t) \in C_1([0,\infty),\mathbb{R}^n)$, $(u(\cdot,t),u_t(\cdot,t)) \in C([0,\infty),C_1[0,1] \times C[0,1])$. Moreover, there is a $\mathscr{K}\mathscr{L}$ function $\overline{\beta}$ such that

$$\Omega(t) \le \overline{\beta}(\Omega(0),t) \tag{27}$$

$$\Omega(t) = |X(t)| + \|u(t)\|_\infty + \|u_t(t)\|_\infty + \|u_x(t)\|_\infty, \tag{28}$$

for all $t \ge 0$.

Stability analysis of the closed-loop system will be derived in the following five steps:

1) Introduction of two transformations to recast the system into a coupled 2×2 hyperbolic PDEs with spatially-varying coefficients is shown in Section III.
2) Removal of the internal PDE states coupling acting on the resulting 2×2 the hyperbolic PDEs via a first backstepping transformation is shown in Section IV-A.
3) The predictor feedback of the equivalent decoupled PDE/nonlinear ODE target system obtained from a second backstepping transformation is introduced in Section IV-B.
4) Stability analysis for the target system is provided in Section V.
5) Proof of stability of the original cascaded system is established in Section VI.

Remark 3: The proof of the well-posedness of the gain kernel PDEs defined in (21)–(24) can be found in [39] which deals with the stabilization of a 2×2 coupled hyperbolic PDE using backstepping technique.

III. From a wave PDE/ODE cascaded system to a coupled 2×2 hyperbolic PDE system with spatially-varying coefficients

In this section we employ two changes of coordinates denoted *Transformation I* and *Transformation II* in order to map system (1)–(4) into a suitable coupled first-order hyperbolic system cascading into a nonlinear ODE.

A. Coupled hyperbolic system: Transformation I

First, we introduce the following change of coordinate

$$\overline{\zeta}(x,t) = \partial_t u(x,t) + \sqrt{v(x)}\,\partial_x u(x,t), \tag{29}$$

$$\overline{\eta}(x,t) = \partial_t u(x,t) - \sqrt{v(x)}\,\partial_x u(x,t), \tag{30}$$

which in reverse is written as

$$\partial_t u(x,t) = \frac{\overline{\zeta}(x,t) + \overline{\eta}(x,t)}{2}, \tag{31}$$

$$\partial_x u(x,t) = \frac{\overline{\zeta}(x,t) - \overline{\eta}(x,t)}{2\sqrt{v(x)}}. \tag{32}$$

Taking the time and spatial derivatives of (29) and (30), we map the original system (2)–(4) into the following coupled hyperbolic PDE cascading with the nonlinear ODE defined in (1)(see Fig. 1).

$$\dot{X} = f(X,u(0,t)) \tag{33}$$

$$\partial_t \overline{\xi}(x,t) = \mathscr{A}(x)\partial_x \overline{\xi}(x,t) + \mathscr{B}_0(x)\overline{\xi}(x,t) \tag{34}$$

$$\partial_t u(0,t) = \overline{\zeta}(0,t) \tag{35}$$

$$\overline{\eta}(0,t) = \overline{\zeta}(0,t) \tag{36}$$

$$\overline{\zeta}(L,t) = \overline{\eta}(L,t) + 2\sqrt{v(L)}\,U(t), \tag{37}$$

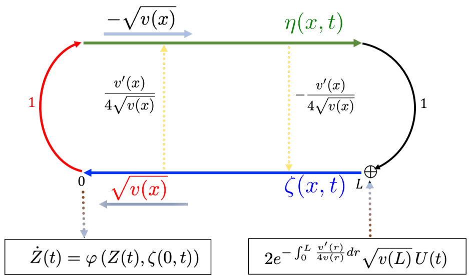

Fig. 1: Equivalent 2×2 coupled hyperbolic PDEs system cascading into a nonlinear ODE

where

$$\bar{\xi}(x,t)=\begin{bmatrix}\bar{\zeta}(x,t)\\ \bar{\eta}(x,t)\end{bmatrix},\mathscr{B}_0(x)=\begin{bmatrix}\frac{-v'(x)}{4\sqrt{v(x)}} & \frac{-v'(x)}{4\sqrt{v(x)}}\\ \frac{v'(x)}{4\sqrt{v(x)}} & \frac{v'(x)}{4\sqrt{v(x)}}\end{bmatrix}, \quad (38)$$

and $\mathscr{A}(x)$ is given by (26).

B. Coupled hyperbolic system: Transformation II

Applying the state transformations

$$\zeta(x,t)=e^{-\int_0^x \frac{v'(r)}{4v(r)}dr}\bar{\zeta}(x,t), \quad (39)$$

$$\eta(x,t)=e^{-\int_0^x \frac{v'(r)}{4v(r)}dr}\bar{\eta}(x,t), \quad (40)$$

system (33)–(37) is rewritten in the following form

$$\dot{Z}(t)=\varphi(Z(t),\zeta(0,t)) \quad (41)$$

$$\partial_t\xi(x,t)=\mathscr{A}(x)\partial_x\xi(x,t)+\mathscr{B}(x)\xi(x,t) \quad (42)$$

$$\zeta(0,t)=\eta(0,t) \quad (43)$$

$$\zeta(L,t)=\eta(L,t)+2e^{-\int_0^L \frac{v'(r)}{4v(r)}dr}\sqrt{v(L)}U(t), \quad (44)$$

where $\xi(x,t)=\begin{bmatrix}\zeta(x,t), & \eta(x,t)\end{bmatrix}^T$, and $\mathscr{A}(x)$ and $\mathscr{B}(x)$ are given by (26).

IV. Backstepping Transformations

A. First step backstepping transformation

In this section, we employ a first backstepping transformation to system (41)–(44) in order to remove the internal PDE state coupling terms of the two propagating waves as shown in Fig. 3:

$$\omega(x,t) = \xi(x,t)-\int_0^x K(x,s)\xi(s,t)ds, \quad (45)$$

for all $0\le x\le L$, $t\ge 0$. The gain kernel matrix $K(x,s)$ is a solution to the kernel equations (21)–(24), where (21) is defined on $\{(x,s): 0\le s\le x\le L\}$. Following [39], it can be verified that (45) has a bounded inverse defined as

$$\xi(x,t) = \omega(x,t)+\int_0^x L(x,s)\omega(s,t)ds, \quad (46)$$

for all $0\le x\le L, t\ge 0$. Moreover, the inverse gain kernel matrix

$$L(x,s)=\begin{bmatrix}L_{11}(x,s) & L_{12}(x,s)\\ L_{21}(x,s) & L_{22}(x,s)\end{bmatrix}\in\mathbb{R}^{2\times 2}, \quad (47)$$

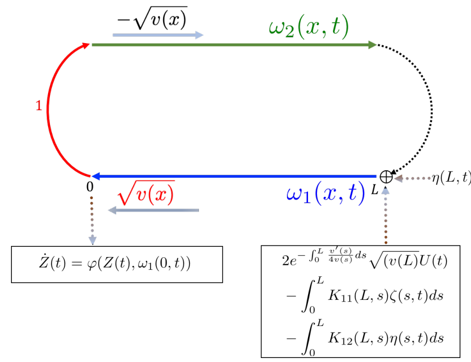

Fig. 2: First-step backstepping transformation removes the internal coupling terms

is the solution of the well-posed kernel equations written below

$$\partial_s L(x,s)\mathscr{A}(x))-\mathscr{A}(x)\partial_x L(x,s) = -\mathscr{B}(x)L(x,s) \quad (48)$$

$$L(x,x)\mathscr{A}(x)-\mathscr{A}(x)L(x,x) = \mathscr{B}(x) \quad (49)$$

$$L_{11}(x,0) = L_{12}(x,0) \quad (50)$$

$$L_{21}(x,0) = L_{22}(x,0), \quad (51)$$

where $\mathscr{A}(x)$ and $\mathscr{B}(x)$ are defined by (26).

Differentiating (45) with respect to time t and space x, it can be straightforwardly established that system (41)–(44) maps into the following decoupled PDE/ODE cascaded system

$$\dot{Z}(t)=\varphi(Z(t),\,\omega_1(0,t)) \quad (52)$$

$$\partial_t\omega(x,t)=\mathscr{A}(x)\,\partial_x\omega(x,t) \quad (53)$$

$$\omega_2(0,t)=\omega_1(0,t) \quad (54)$$

$$\begin{aligned}\omega_1(L,t)=&\ \eta(L,t)+2e^{-\int_0^L \frac{v'(r)}{4v(r)}dr}\sqrt{v(L)}U(t)\\ &-\int_0^L K_{11}(L,s)\zeta(s,t)ds\\ &-\int_0^L K_{12}(L,s)\eta(s,t)ds,\end{aligned} \quad (55)$$

for $0\le x\le L, t\ge 0$, and $\omega(x,t)=[\omega_1(x,t),\omega_2(x,t)]^T$ if the gain kernel matrix satisfies (21)–(24). The ODE dynamics is driven by φ defined in (7).

B. Second-step backstepping transformation

From a nominal controller $\kappa:\mathbb{R}^n\to\mathbb{R}$ that globally asymptotically stabilizes the original delay-free nonlinear ODE, we define the following function

$$\mu(\chi) = -c_1(\chi_2-\kappa(\chi_1))+\frac{\partial\kappa(\chi_1)}{\partial\chi_1}f(\chi_1,\chi_2), \quad (56)$$

where $c_1>0$ is an arbitrary gain constant and $\chi=[\chi_1,\chi_2]\in\mathbb{R}^n\times\mathbb{R}$. Denote the vector functions $p(x,t)$ and $q(x,t)$ as

$$p(x,t)=Z(t)+\int_0^x \frac{\varphi(p(y,t),\,\omega_1(y,t))}{\sqrt{v(y)}}dy, \quad (57)$$

where $p(x,t)=[p_1(x,t),p_2(x,t)]^T$, with the initial condition

$$p(x,0)=Z(0)+\int_0^x \frac{\varphi(p(y,0),\,\omega_1(y,0))}{\sqrt{v(0)}}dy, \quad (58)$$

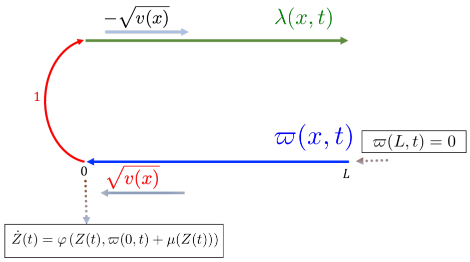

Fig. 3: Second-step backsteping transformation removes the boundary coupling at $x = L$, and designs a compensator for the resulting cascaded system

and

$$q(x,t) = Z(t) - \int_0^x \frac{\varphi(q(y,t),\,\omega_2(y,t))}{\sqrt{v(y)}}dy, \tag{59}$$

where $q(x,t) = [q_1(x,t), q_2(x,t)]^T$, with the initial condition

$$q(x,0) \quad = Z(0) - \textstyle\int_0^x \frac{\varphi(q(y,0),\omega_2(y,0))}{\sqrt{v(0)}}dy. \tag{60}$$

Lemma 1 (Second-Step Backstepping Transform): The following backstepping transformations

$$\varpi(x,t) \quad = \quad \omega_1(x,t) - \mu(p(x,t)), \tag{61}$$

$$\lambda(x,t) \quad = \quad \omega_2(x,t) - \mu(q(x,t)), \tag{62}$$

where μ is defined in (56), and $p(x,t), q(x,t)$ are given as (57), (59), respectively, and $U(t)$ is

$$U(t) = \frac{e^{\int_0^L \frac{v'(s)}{4v(s)}ds}}{2\sqrt{v(L)}}\Big(\mu(p(L,t)) - \eta(L,t) + \int_0^L \big(K_{11}(L,s)\zeta(s,t) + K_{12}(L,s)\eta(s,t)\big)ds\Big), \tag{63}$$

map system (52)–(55) into the target system

$$\dot{Z}(t) = \varphi(Z(t), \varpi(0,t) + \mu(Z(t))) \tag{64}$$

$$\partial_t \varpi(x,t) = \sqrt{v(x)}\,\partial_x \varpi(x,t) \tag{65}$$

$$\partial_t \lambda(x,t) = -\sqrt{v(x)}\,\partial_x \lambda(x,t) \tag{66}$$

$$\lambda(0,t) = \varpi(0,t) \tag{67}$$

$$\varpi(L,t) = 0. \tag{68}$$

The schematic of the resulting target system is depicted in Fig. 3.

Proof. It is easy to obtain (64), (67), (68). We will prove relation (65). Differentiating (57) with respect to t and x, we get

$$\partial_t p(x,t) = \varphi(Z(t),\,\omega_1(0,t)) + \int_0^x \frac{\partial_p \varphi(p(y,t),\omega_1(y,t))\partial_t p(y,t)}{\sqrt{v(y)}}dy + \int_0^x \frac{\partial_{\omega_1}\varphi(p(y,t),\,\omega_1(y,t))\,\partial_t \omega_1(y,t)}{\sqrt{v(y)}}dy, \tag{69}$$

and

$$\sqrt{v(x)}\partial_x p(x,t) = \int_0^x \partial_p \varphi(p(y,t),\omega_1(p(y,t)))\partial_y p(y,t)dy + \int_0^x \partial_{\omega_1}\varphi(p(y,t),\,\omega_1(p(y,t)))\,\partial_y \omega_1(y,t)dy + \varphi(Z(t),\,\omega_1(0,t)), \tag{70}$$

respectively. Defining $H(x,t) = \partial_t p(x,t) - \sqrt{v(x)}\partial_x p(x,t)$, and combining (69) and (70), and by (53), we arrive at

$$H(x,t) = \int_0^x \frac{\partial_p \varphi(p(y,t),\omega_1(y,t))H(y,t)}{\sqrt{v(y)}}dy. \tag{71}$$

Differentiating (71) with respect to x, we have

$$\partial_x H(x,t) = \frac{\partial_p \varphi(p(x,t),\omega_1(x,t))H(x,t)}{\sqrt{v(x)}}, \tag{72}$$

and $H(0,t) = 0$, which implies that $H(x,t) = 0$, for all $x \in [0,L]$. Hence, it is clear that

$$\partial_t p(x,t) = \sqrt{v(x)}\partial_x p(x,t). \tag{73}$$

Taking the time and the spatial derivatives of (61), and from (53) and (73), we obtain (65). Relation (66) can be deduced similarly, which completes the proof.

Remark 4: Using (39)*, it is easy to show that* (63) *can be expressed as follows:*

$$U(t) = \frac{e^{\int_0^L \frac{v'(r)}{4v(r)}ds}}{2\sqrt{v(L)}}\mu(p(L,t)) - \frac{1}{2\sqrt{v(L)}}\bar{\eta}(L,t) + \frac{1}{2\sqrt{v(L)}}\int_0^L K_{11}(L,s)\bar{\zeta}(s,t))ds + \frac{1}{2\sqrt{v(L)}}\int_0^L K_{12}(L,s)\bar{\eta}(s,t)ds, \tag{74}$$

Substituting (29)*,* (30)*, and* (56) *into* (74) *gives the equivalent control action* $U(t)$ *defined in* (16)*. In addition, it can be deduced that* $p(x,t)$ *in* (57) *is equal to* $[p_1(x,t), p_2(x,t)]^T$*, where* $p_1(x,t), p_2(x,t)$ *are given by* (17)*,* (18)*, respectively.*

Define the vector functions $\pi(x,t)$ and $\iota(x,t)$ as

$$\pi(x,t) = Z(t) + \int_0^x \frac{\varphi(\pi(y,t), \varpi(y,t) + \mu(\pi(y,t)))}{\sqrt{v(y)}}dy, \tag{75}$$

where $\pi(x,t) = [\pi_1(x,t), \pi_2(x,t)]^T$, with the initial condition

$$\pi(x,0) = Z(0) + \int_0^x \frac{\varphi(\pi(y,0), \varpi(y,0) + \mu(\pi(y,0)))}{\sqrt{v(y)}}dy, \tag{76}$$

and

$$\iota(x,t) = Z(t) - \int_0^x \frac{\varphi(\iota(y,t),\,\lambda(y,t) + \mu(\iota(y,t)))}{\sqrt{v(y)}}dy, \tag{77}$$

where $\iota(x,t) = [\iota_1(x,t), \iota_2(x,t)]^T$ with the initial condition

$$\iota(x,0) = Z(0) - \int_0^x \frac{\varphi(\iota(y,0),\,\lambda(y,0) + \mu(\iota(y,0)))}{\sqrt{v(y)}}dy, \tag{78}$$

where $\varpi, \lambda,\ \mu$ are defined in (61), (62), (56), respectively.

Inverse Backstepping Transforms: The inverse backstepping transformations of ϖ,λ are defined as

$$\omega_1(x,t)=\varpi(x,t)+\mu(\pi(x,t)), \tag{79}$$

$$\omega_2(x,t)=\lambda(x,t)+\mu(\iota(x,t)), \tag{80}$$

where $\pi(x,t)$, $\iota(x,t)$, $0\le x\le L$, $t\ge 0$, are given as (75), (77), respectively.

The inverse backstepping transformations (79), (80), and the control law (63) transform the target system (64)–(68) into system (52)–(55) and the proof can be derived from straightforward computations.

V. Stability analysis of the target system

Lemma 2 (Stability of the Target System): Consider system (64)–(68), under Assumptions 1 and 2, there exists a class $\mathscr{K}\mathscr{L}$ function β, such that

$$\begin{aligned}&|Z(t)|+\|\varpi(t)\|_\infty+\|\lambda(t)\|_\infty\\ &\le\beta(|Z(0)|+\|\varpi(0)\|_\infty+\|\lambda(0)\|_\infty,t),\end{aligned} \tag{81}$$

for all $t\ge 0$.

Proof. We introduce a new variable $z(x,t), x\in[-L,L]$ such that

$$z(x,t)=\begin{cases}\varpi(x,t), & \text{for all } x\in[0,L],\\ \lambda(-x,t), & \text{for all } x\in[-L,0].\end{cases} \tag{82}$$

Let $\Gamma_{g,n}(t)$ denote the following norm

$$\Gamma_{g,n}(t)=\int_{-L}^{L}e^{2ng(L+x)}z(x,t)^{2n}dx, \tag{83}$$

where $g>0$ is determined later and n is any positive integer. Using integration by parts, the derivative of $\Gamma_{g,n}(t)$ satisfies

$$\begin{aligned}\dot\Gamma_{g,n}(t)\le&-\int_{-L}^{0}2n\left(g\sqrt{v(-x)}-\frac{v'(-x)}{4n\sqrt{v(-x)}}\right)\\ &\times e^{2ng(L+x)}z(x,t)^{2n}dx\\ &-\int_{0}^{L}2n\left(g\sqrt{v(x)}+\frac{v'(x)}{4n\sqrt{v(x)}}\right)e^{2ng(L+x)}z(x,t)^{2n}dx.\end{aligned} \tag{84}$$

Under Assumption 2, $v(x)$ is continuously differentiable and positive for all $x\in[0,L]$, denote

$$\pi_0=\sup_{x\in[0,L]}\frac{v'(x)}{\sqrt{v(x)}},\quad \pi_1=\inf_{x\in[0,L]}\frac{v'(x)}{\sqrt{v(x)}}. \tag{85}$$

Note $n\ge 1$, we have

$$g\sqrt{v(-x)}-\frac{v'(-x)}{4n\sqrt{v(-x)}}\ge\min\left\{g\sqrt{\underline{\upsilon}}-\frac{\pi_0}{4},\ g\sqrt{\underline{\upsilon}}\right\} \tag{86}$$

for all $x\in[-L,0]$, and

$$g\sqrt{v(x)}+\frac{v'(x)}{4n\sqrt{v(x)}}\ge\min\left\{g\sqrt{\underline{\upsilon}}+\frac{\pi_1}{4},\ g\sqrt{\underline{\upsilon}}\right\}, \tag{87}$$

for all $x\in[0,L]$, and $\underline{\upsilon}$ is given by (5). Choose

$$g>\max\left\{\frac{\pi_0}{4\sqrt{\underline{\upsilon}}},\ \frac{-\pi_1}{4\sqrt{\underline{\upsilon}}}\right\}, \tag{88}$$

we get

$$\dot\Gamma_{g,n}(t)\le-2n\pi_2\Gamma_{g,n}(t),\quad \text{for}\quad t\ge 0, \tag{89}$$

with $\pi_2=\min\{\min\{g\sqrt{\underline{\upsilon}}-\frac{\pi_0}{4},\ g\sqrt{\underline{\upsilon}}\},\min\{g\sqrt{\underline{\upsilon}}+\frac{\pi_1}{4},\ g\sqrt{\underline{\upsilon}}\}\}$. Using Assumption 2, from (89), it is easy to deduce that (81) holds.

VI. Stability of the original wave PDE/nonlinear ODE cascaded system

To establish stability proof of the closed-loop system (1)–(4), (16)–(18), we show the boundedness of predictors, first. Proofs of Lemmas 3–8 are established using systematic developments already presented in [10] and [13].

Lemma 3 (Bound on Forward Predictor): Under Assumptions 1 and 3, there exists a class $\mathscr{K}_\infty$ function ρ_1 such that the following holds:

$$\sup_{0\le x\le L}|p(x,t)|\le\rho_1(|Z(t)|+\|\omega_1(t)\|_\infty). \tag{90}$$

Lemma 4 (Bound on Backward Predictor): Under Assumptions 1 and 4, there exists a class $\mathscr{K}_\infty$ function ρ_2 such that the following holds:

$$\sup_{0\le x\le L}|q(x,t)|\le\rho_2(|Z(t)|+\|\omega_2(t)\|_\infty). \tag{91}$$

Lemma 5 (Bound on Extended Forward State Predictor): Under Assumptions 1 and 2, there exists a class $\mathscr{K}_\infty$ function ρ_3 such that the following holds:

$$\sup_{0\le x\le L}|\pi(x,t)|\le\rho_3(|Z(t)|+\|\varpi(t)\|_\infty). \tag{92}$$

Lemma 6 (Bound on Extended Backward State Predictor): Under Assumptions 1 and 4, there exists a class $\mathscr{K}_\infty$ function ρ_4 such that the following holds:

$$\sup_{0\le x\le L}|\iota(x,t)|\le\rho_4(|Z(t)|+\|\lambda(t)\|_\infty). \tag{93}$$

Next, we show equivalence of norms of original and target PDE states in Lemma 7 and Lemma 8.

Lemma 7: **(Original PDE State Bounded by Target PDE State)** Under Assumptions 2 and 4, consider system (64)–(68), and output maps are (79), (80), then there exists a class $\mathscr{K}_\infty$ function γ_2 such that the following holds:

$$\begin{aligned}&|Z(t)|+\|\omega_1(t)\|_\infty+\|\omega_2(t)\|_\infty\\ &\le\gamma_2(|Z(t)|+\|\varpi(t)\|_\infty+\|\lambda(t)\|_\infty).\end{aligned} \tag{94}$$

Lemma 8: **(Target PDE State Bounded by Original PDE State)** Under Assumptions 2 and 4, consider system (52)–(55), and output maps are (61), (62), then there exists a class $\mathscr{K}_\infty$ function γ_3 such that the following holds:

$$\begin{aligned}&|Z(t)|+\|\varpi(t)\|_\infty+\|\lambda(t)\|_\infty\\ &\le\gamma_3(|Z(t)|+\|\omega_1(t)\|_\infty+\|\omega_2(t)\|_\infty).\end{aligned} \tag{95}$$

Proof of Theorem 1. Using Lemmas 2, 7 and 8, with the help of (45), (46), we get

$$
\begin{aligned}
&|Z(t)|+\|\xi(t)\|_\infty\\
&\le |Z(t)|+(1+\overline{L})\|\omega(t)\|_\infty\\
&\le (1+\overline{L})\gamma_2(|Z(t)|+\|\varpi(t)\|_\infty+\|\lambda(t)\|_\infty)\\
&\le (1+\overline{L})\gamma_2(\beta(|Z(0)|+\|\varpi(0)\|_\infty+\|\lambda(0)\|_\infty,t))\\
&\le (1+\overline{L})\gamma_2(\beta(\gamma_3(|Z(0)|+\|\omega_1(0)\|_\infty+\|\omega_2(0)\|_\infty),t))\\
&\le (1+\overline{L})\gamma_2(\beta(\gamma_3(\sqrt{2}(|Z(0)|+\|\omega(0)\|_\infty),t))\\
&\le (1+\overline{L})\gamma_2(\beta(\gamma_3(\sqrt{2}(1+\overline{K})(|Z(0)|+\|\xi(0)\|_\infty),t)),
\end{aligned} \tag{96}
$$

where $\overline{L}=\max_{(x,y)\in[0,L]\times[0,L]}|L(x,y)|$, $\overline{K}=\max_{(x,y)\in[0,L]\times[0,L]}|K(x,y)|$.

Using (7), (29)–(32), (39), (40), and (96), we obtain the following estimate

$$
\begin{aligned}
&|X(t)|+|u(0,t)|+\|\partial_t u(t)\|_\infty+\|\partial_x u(t)\|_\infty\\
&\le \sqrt{2}|Z(t)|+\tfrac{\sqrt{2}}{2}(1+\tfrac{1}{\sqrt{\underline{\upsilon}}})\sqrt[4]{\tfrac{\overline{\upsilon}}{\underline{\upsilon}}}\|\xi(t)\|_\infty\\
&\le \max\{\sqrt{2},\tfrac{\sqrt{2}}{2}(1+\tfrac{1}{\sqrt{\underline{\upsilon}}})\sqrt[4]{\tfrac{\overline{\upsilon}}{\underline{\upsilon}}}\}(|Z(t)|+\|\xi(t)\|_\infty)\\
&\le (1+\overline{L})\max\{\sqrt{2},\tfrac{\sqrt{2}}{2}(1+\tfrac{1}{\sqrt{\underline{\upsilon}}})\sqrt[4]{\tfrac{\overline{\upsilon}}{\underline{\upsilon}}}\}\\
&\times\gamma_2(\beta(\gamma_3(\sqrt{2}(1+\overline{K})(|Z(0)|+\|\xi(0)\|_\infty),t))\\
&\le (1+\overline{L})\max\{\sqrt{2},\tfrac{\sqrt{2}}{2}(1+\tfrac{1}{\sqrt{\underline{\upsilon}}})\sqrt[4]{\tfrac{\overline{\upsilon}}{\underline{\upsilon}}}\}\\
&\times\gamma_2(\beta(\gamma_3(\sqrt{2}(1+\overline{K})(|X(0)|+\|u(0)\|_\infty+2\sqrt[4]{\tfrac{\overline{\upsilon}}{\underline{\upsilon}}}(1+\sqrt{\overline{\upsilon}})\\
&\times(\|\partial_t u(0)\|_\infty+\|\partial_x u(0)\|_\infty)),t)).
\end{aligned} \tag{97}
$$

Finally defining a class $\mathscr{K}_\infty$ function $\overline{\beta}(s,t)=(1+\overline{L})\max\{\sqrt{2},\frac{\sqrt{2}}{2}(1+\frac{1}{\sqrt{\underline{\upsilon}}})\sqrt[4]{\frac{\overline{\upsilon}}{\underline{\upsilon}}}\}\gamma_2(\beta(\gamma_3(\sqrt{2}(1+\overline{K})+2\sqrt[4]{\frac{\overline{\upsilon}}{\underline{\upsilon}}}(1+\sqrt{\overline{\upsilon}})s),t))$, we get (27).

Following [11], it can be proved that under Assumptions 1–4 and $u(\cdot,0)\in C_1[0,1]$, $u_t(\cdot,0)\in C[0,1]$, which is compatible with the feedback law (16)–(18), the closed-loop system has a unique solution $X(t)\in C_1([0,\infty),\mathbb{R}^n)$, $(u(\cdot,t),u_t(\cdot,t))\in C([0,\infty),C_1[0,L]\times C[0,L])$.

VII. Simulation results

We consider a cable that is made of dynamic material, and a load that is the benchmark system modeled as a point mass M is attached to the lower end of the cable. Small deflections of the cable denoted $u(x,t)$ obeys the planar motion governed by (2)–(4), where $\partial_x u(1,t)$ of the cable at its upper end serves as a control input, and $v(x)=\sqrt{g\rho\,(1+\sin(x))+Mg}$ is the force in the cable, and $X(t)=[X_1,X_2,X_3]^T$ is the state of the load which satisfies

$$\dot{X}_1=X_2+X_3^2 \tag{98}$$
$$\dot{X}_2=X_3 \tag{99}$$
$$\dot{X}_3=-X_2-2X_3+u(0,t). \tag{100}$$

Here, ρ is the material's density and g is the gravity constant. Following [40], a nominal design for system (98)–(100) is given as

$$
\begin{aligned}
\kappa(X)=&-X_3-(X_1+2X_2+X_3+0.25X_2^2\\
&+0.25X_3^2)(1+0.5X_3).
\end{aligned} \tag{101}
$$

The control law for system (2)–(4) cascading with (98)–(100) is given by (16)–(18). A simulation study is performed

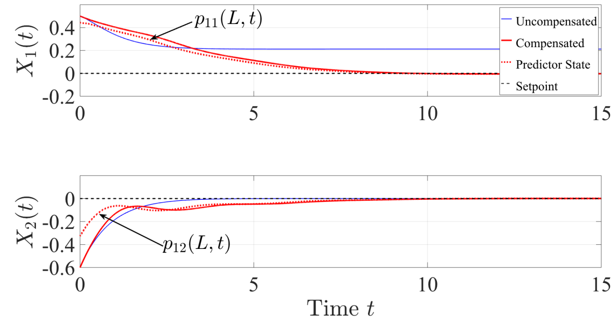

Fig. 4: Response of $(X_1(t),X_2(t))$ under the proposed control (solid red line) and with uncompensated control (101) (blue line), state prediction $(p_{11}(L,t),p_{12}(L,t))$ (dashdot red line)

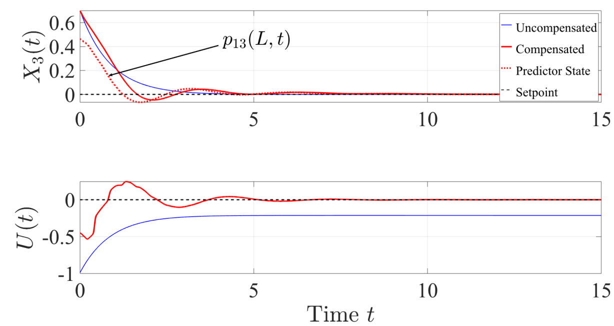

Fig. 5: Dynamics of $(X_3(t),U(t))$ under the proposed control (solid red line) and with uncompensated control (101) (blue line), state prediction $p_{13}(L,t)$ (dashdot red line)

with $L=1m$, $M=1kg$, $\rho=2kg/m$, and $g=9.81m/s^2$, for the initial values of the ODE states given as $X_1(0)=0.5$, $X_2(0)=-0.6$, $X_3(0)=0.7$ and $u_x(x,0)=0$, $u_t(x,0)=0.5$ for $x\in[0,1]$ and the gain parameter $c_1=1$. Responses of the states (X_1,X_2,X_3) together with the predictor states $(p_{11}(L,t),p_{12}(L,t),p_{13}(L,t))$ under the proposed control law and the nominal control action without wave actuator dynamics compensation are shown in Fig. 4–Fig. 5. It is clear that the designed predictor enables to compute the future values of the real states and the designed predictor-feedback controller stabilizes the system at the setpoint. However, the uncompensated control action (101) cannot achieve stabilization of the state X_1 to the desired setpoint and leads to a bounded dynamics as depicted in Fig.4. The actuator dynamics for the compensated case is depicted in Fig.6, which confirms the pertinence of the proposed control law.

VIII. Conclusion

We consider boundary control of nonlinear ODE/wave PDE cascaded systems with spatially-varying propagation speed. A nonlinear backstepping compensator is designed such that the closed-loop system is globally asymptotically stable and the stability proof is established based on a Lyapunov-like

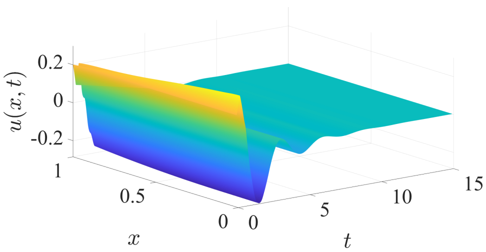

Fig. 6: Response of wave PDE dynamics under the proposed control

argument. The proposed design is illustrated by the benchmark system controlled via a cable. The generalization of the result to actuator dynamics governed by an arbitrary number of coupled linear hyperbolic PDEs [17], the extension to drilling systems' stabilization problems and to the particular case $v(x) = 0$ will be considered in our future works.

References

[1] N. Bekiaris-Liberis, and M. Krstic, Nonlinear Control Under Nonconstant Delays. SIAM, 2013.

[2] N. Bekiaris-Liberis, and M. Krstic, "Compensation of state-dependent input delay for nonlinear systems," IEEE Transactions on Automatic Control, vol. 58, pp. 275-289, 2013.

[3] N. Bekiaris-Liberis, and M. Krstic, "Nonlinear control under delays that depend on delayed states," European Journal of Control, vol.19, pp. 389-398, 2013.

[4] M. Diagne, N. Bekiaris-Liberis, and M. Krstic, "Time-and state-dependent input delay-compensated bang-bang control of a screw extruder for 3D printing," International Journal of Robust and Nonlinear Control, vol. 27, pp. 3727–3757, 2017.

[5] D. Bresch-Pietri, J. Chauvin, and N. Petit, "Invoking halanay inequality to conclude on closed-loop stability of processes with input-varying delay," in IFAC workshop on time delay systems, Boston, USA, 2012.

[6] D. Bresch-Pietri, J. Chauvin, and N. Petit, "Prediction-based feedback control of a class of processes with input-varying delay," in American control conference, Canada, Montreal, 2012.

[7] D. Bresch-Pietri, J. Chauvin, and N. Petit, "Prediction-based stabilization of linear systems subject to input-dependent input delay of integral-type," IEEE Transaction on Automatic Control, vol. 59, pp. 2385-2399, 2014.

[8] D. Bresch-Pietri, and M. Krstic, "Output-feedback adaptive control of a wave PDE with boundary anti-damping," Automatica, vol. 50, pp. 1407-1415, 2014.

[9] N. Bekiaris-Liberis, and M. Krstic, "Compensation of wave actuator dynamics for nonlinear systems," IEEE Transaction on Automatic Control, vol. 59, pp. 1555-1570, 2014.

[10] X. Cai, and M. Krstic, "Nonlinear control under wave actuator dynamics with time-and state-dependent moving boundary," International Journal of Robust and Nonlinear Control, vol. 25, pp. 222-253, 2015.

[11] X. Cai, and M. Krstic, "Nonlinear stabilization through wave PDE dynamics with a moving uncontrolled boundary," Automatica, vol. 68, pp. 27-38, 2016.

[12] M. Diagne, N. Bekiaris-Liberis, A. Otto, and M. Krstic, "Compensation of input delay that depends on delayed input," Automatica, vol. 85, pp. 362-373, 2017.

[13] M. Diagne, N. Bekiaris-Liberis, A. Otto, and M. Krstic, "Control of transport PDE/nonlinear ODE cascades with state-dependent propagation speed," IEEE Transactions on Automatic Control, vol. 62, pp. 6278-6293, 2017.

[14] N. Bekiaris-Liberis, and M. Krstic, "Compensation of actuator dynamics governed by quasilinear hyperbolic PDEs," Automatica, vol. 92, pp. 29-40, 2018.

[15] M. Krstic, and A. Smyshlyaev, "Backstepping boundary control for first-order hyperbolic PDEs and application to systems with actuator and sensor delays," System and Control Letters, vol. 57, pp. 750-758, 2008.

[16] F. Di Meglio, F. Bribiesca Argomedo, L. Hu, and M. Krstic, "Stabilization of coupled linear heterodirectional hyperbolic PDE-ODE systems," Automatica, vol. 87, pp. 281-289, 2018.

[17] F. Di Meglio, R. Vazquez, and M. Krstic, "Stabilization of a system of $n+1$ coupled first-order hyperbolic linear PDEs with a single boundary input," IEEE Transactions on Automatic Control, vol. 58, pp. 3097-3111, 2013.

[18] J. Deutscher, N. Gehring, and R. Kern, "Backstepping control of linear 2×2 hyperbolic systems with dynamic boundary conditions," In IFAC world congress. Toulouse, France, 2017.

[19] X. Cai, N. Bekiaris-Liberis, and M. Krstic, "Input-to-state stability and inverse optimality of linear time-varying-delay predictor feedbacks," IEEE Transactions on Automatic Control, vol. 63, pp. 233-240, 2018.

[20] X. Cai, N. Bekiaris-Liberis, and M. Krstic, "Input-to-state stability and inverse optimality of predictor feedback for multi-input linear systems," Automatica, vol. 103, pp. 549-557, 2019.

[21] X. Cai, C. Lin, L. Liu, and X. Zhan, "Inverse optimal control for strict-feedforward nonlinear systems with input delays," International Journal of Robust and Nonlinear Control, vol.28, pp. 2976-2995, 2018.

[22] X. Cai, L. Meng, W. Zhang, and L. Liu, "Inverse optimal design of input-to-state stabilisation for affine nonlinear systems with input delays," International Journal of Systems Science, vol. 49, pp. 832-846, 2018.

[23] A. Otto, and G. Radons, "Application of spindle speed variation for chatter suppression in turning," CIRP J. Manuf. Sci. Technol., vol. 6, pp. 102-109, 2013.

[24] M. Herty, J. Lebacque, and S. Moutari, "A novel model for intersections of vehicular traffic flow," Networks and Heterogenous Media, vol. 4, pp. 813-826, 2009.

[25] D. Bresch-Pietri, and K. Coulon, "Prediction-based control of moisture in a convective flow," in Proc. IEEE Eur. Control Conference, Linz, Austria, 2015.

[26] L. Guzzella, and C. Onder, Introduction to Modeling and Control of Internal Combustion Engine Systems, New York: Springer, 2009.

[27] M. Jankovic, and S. Magner, "Disturbance attenuation in time-delay systems- a case study on engine air-fuel ratio control," in American control conference, San Francisco, California, 2011.

[28] M. Chebre, Y. Creff, and N. Petit, "Feedback control and optimization for the production of commercial fuels by blending," J. Process Control, vol. 20, pp. 441-451, 2010.

[29] M. Krstic, Delay Compensation for Nonlinear, Adaptive, and PDE Systems, Boston: Birkhauser, 2009.

[30] C. Sagert, F. Di Meglio, M.Krstic, and P. Rouchon, "Backstepping and flatness approaches for stabilization of the stick-slip phenomenon for drilling," in Proceedings of IFAC symposium on system, structure and control. TDS Grenoble, France, 2013.

[31] J., Wang, S. X. Tang, and M. Krstic, "Adaptive output-feedback control of torsional vibration in off-shore rotary oil drilling systems," Automatica, vol 111, https://doi.org/10.1016/j.automatica.2019.108640, 2020.

[32] B. Saldivar, S. Mondie, S. I. Niculescu, H. Mounier, and I. Boussaada, "A control oriented guided tour in oilwell drilling vibration modeling," Annual Reviews in Control, vol. 42, pp. 100-113, 2016.

[33] F. Di Meglio, and U. J. F. Aarsnes, " A distributed parameter systems view of control problems in drilling," 2nd IFAC Workshop on Automatic Control in Offshore Oil and Gas Production OOGP, vol. 48(6), pp. 272-278, 2015.

[34] I. Boussaada, H. Mounier, S.I. Niculescu, S. I., and A. Cela, " Analysis of drilling vibrations: A time-delay system approach," in Proceedings of the Mediterranean Conference on Control & Automation (MED), pp. 610-614, 2012.

[35] B. Saldivar, S. Mondie, J. C. A. Vilchis, "The control of drilling vibrattions: A coupled PDE-ODE modeling approach," Int. J. Appl. Math. Comput. Sci., 2016, vol. 26(2), pp. 335-349, 2016.

[36] B. d'Andrea-Novel, J.M. Coron, "Exponential stabilization of an overhead crane with flexible cable via a back-stepping approach," vol. 36, pp. 587-593, 2000.

[37] J. Wang, Y. Pi, and M. Krstic, "Balancing and suppression of oscillations of tension and cage in dual-cable mining elevators," Automatica, vol. 98, pp. 223-238, 2018.

[38] J. Wang, and M. Krstic, "Vibration suppression for coupled wave PDEs in deep-sea construction," arXiv preprint arXiv: 1911.08455.

[39] R. Vazquez, M. Krstic, and J. Coron, "Backstepping boundary stabilization and state estimation of a 2×2 linear hyperbolic system," in IEEE conference on decision and control and European control conference, Orlando, FL, USA, 2011.

[40] R. Sepulchre, M.Jankovic, and P. Kokotovic, Constructive Nonlinear Control, London: Springer-Verlag, 1996.